De Rham Cohomology and the de Rham Theorem

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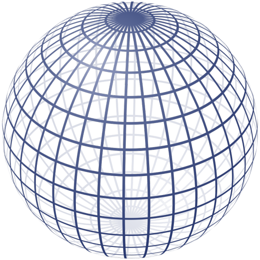
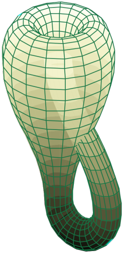
Differential Manifolds

- An *n-manifold* M is a space that "looks like $\mathbb{R}^n$" if you zoom in enough.
- Formally, for any point $x \in M$, there is a neighborhood $U \subseteq M$ containing x in which U is homeomorphic to $\mathbb{R}^n$.
- A differential manifold is a manifold with certain properties that allow us to do calculus over it. Some examples include:
    - The n-sphere S^n
    - Smooth curves and surfaces
    - The torus
    - The Klein bottle





Differential Forms

We have seen in Calc III that $\int_C \vec{F} \cdot d\vec{r} = \int F_1 dx + F_2 dy + F_3 dz$, where $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field being integrated over a curve. This can be generalized for differential manifolds in the form of **differential forms**.

Definition

In the variables $x_1, \ldots, x_n$ a differential 1-form is something of the form $\omega = \sum f_i(x_1, \ldots, x_n) dx_i$, where each f_i is required to be a smooth function.

Definition

A differential k-form is something of the form $\omega = \sum f_{i_1,\ldots,i_k}(x_1, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where "$\wedge$" is the wedge product in multilinear algebra, given by $\omega \wedge \eta = \frac{1}{2}(\omega \otimes \eta) - \frac{1}{2}(\eta \otimes \omega)$. (Note that $\omega \wedge \omega = 0$.)

$\Omega^k(M)$ and the Exterior Derivative

Differential forms end up forming a sequence of abelian groups. We denote the abelian group of all k-forms over a manifold M as $\Omega^k(M)$. We also let $\Omega^0(M)$ be the space of all real-valued smooth functions on M. Thus, this gives us a "cochain complex":

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

where $d^2 = 0$ and d is the exterior derivative described as follows:

Definition

The **exterior derivative** is a homomorphism $d : \Omega^k(M) \to \Omega^{k+1}(M)$, where for $\omega = f \; dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, we have that $d\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial x_{i_j}} \; dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. This then extends linearly to all k-forms.

De Rham Cohomology

There are two kinds of forms that we want to examine:

- A k-form ω is *closed* if $d\omega = 0$. Note that this is the same as saying that $\omega \in \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$.
- A k-form is *exact* if there exists $\eta \in \Omega^{k-1}(M)$ such that $d\eta = \omega$. Note that this is the same as saying that $\omega \in \operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$.

To study when closed forms are exact, we can look at the **cohomology groups** $H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$. These are well defined quotient groups, since the following two facts hold:

1. $d^2 = 0$, and so $\operatorname{im} \subseteq \ker$,
2. $\Omega^k(M)$ is abelian, so $\operatorname{im} \trianglelefteq \ker$.

Integration of Forms

Differential forms can be thought of as the "things we integrate". Thus, we must define how forms can be integrated.

Definition

Let $\omega = f(x_1, \ldots, x_n)\ dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ be an n-form on $U \subset \mathbb{R}^n$. Then, $\int_U \omega = \int_U f(x_1, \ldots, x_n)\ dx_1 dx_2 \cdots dx_n$.

Definition

Let $\varphi : U \subseteq M \to V \subseteq \mathbb{R}^n$ be a homeomorphism and ω an n-form on M. Then, $(\varphi^{-1})^*\omega = (f \circ \varphi^{-1})\ d(x_1 \circ \varphi^{-1}) \wedge \cdots \wedge d(x_n \circ \varphi^{-1})$ is an n-form on $\mathbb{R}^n$ and we define $\int_M \omega = \int_{\mathbb{R}^n} (\varphi^{-1})^*\omega$.

Stokes' Theorem

In Calc III, we see the following theorems:

- Stokes' Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$,
- Divergence Theorem: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\ dV$,
- Green's Theorem: $\int_C P\ dx + Q\ dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\ dA$.

As it turns out, these are all cases of the more generalized Stokes' Theorem.

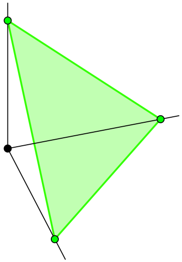
Theorem (Stokes' Theorem)

If M is an oriented n-manifold with boundary ∂M and ω is an $(n-1)$-form on M, then $\int_M d\omega = \int_{\partial M} \omega$.

Singular Simplices

Let M be an n-manifold. Let $e_0, e_1, \ldots, e_n$ be the canonical basis vectors of $\mathbb{R}^{n+1}$.

- The standard n-simplex Δ_n is defined as $\Delta_n = \{x = \sum_{i=0}^{n} \lambda_i e_i \mid \sum \lambda_i = 1,\ 0 \leq \lambda_i \leq 1\}$. This represents the n-dimensional "triangle" embedded in $n+1$ dimensions. Shown below is the standard 2-simplex.
- A singular k-simplex is a continuous map $\sigma : \Delta_k \to M$.



$C_k(M)$ and $\mathrm{Hom}(C_k(M), \mathbb{R})$

- A k-chain c is a finite sum $c = \sum n_i \sigma_i$ for $n_i \in \mathbb{N}$.
- The abelian group $C_k(M)$ is defined as the group of all such chains. This is the additive group generated by all possible k-simplices.
- These chain groups form a chain complex with the boundary operator $\partial : C_k(M) \to C_{k-1}(M)$, given by the actual boundary of the k-simplex Δ_k.
- We can apply the "contravariant functor" $\mathrm{Hom}(-, \mathbb{R})$ to this chain complex, which has the effect of reversing the arrows. This process is analogous to taking the dual space in linear algebra.

$C_k(M)$ and $\mathrm{Hom}(C_k(M),\mathbb{R})$

$$\begin{tikzcd}
0 & & 0 \arrow[d, "\delta"'] \\
C_0(M) \arrow[u, "\partial"'] & & \mathrm{Hom}(C_0(M),\mathbb{R}) \arrow[d, "\delta"'] \\
C_1(M) \arrow[u, "\partial"'] \arrow[rr, "\mathrm{Hom}(-{,}\mathbb{R})"] & & \mathrm{Hom}(C_1(M),\mathbb{R}) \arrow[d, "\delta"'] \\
C_2(M) \arrow[u, "\partial"'] & & \mathrm{Hom}(C_2(M),\mathbb{R}) \arrow[d, "\delta"'] \\
\vdots \arrow[u, "\partial"'] & & \vdots
\end{tikzcd}$$

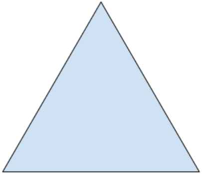
We denote the Hom groups $C^k(M;\mathbb{R}) := \mathrm{Hom}(C_k(M),\mathbb{R}))$.
Also, $\delta : C^k(M;\mathbb{R}) \to C^{k+1}(M;\mathbb{R})$ represents the transpose of ∂.

Singular Cohomology

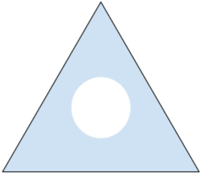
Just like with $\{\Omega^n(M)\}$, we can look at the cohomology groups of $\{C^n(M;\mathbb{R})\}$, defined the same way as

$$H^n(M;\mathbb{R}) = \ker\delta/\text{im }\delta.$$

These groups measure when a chain c is a *cycle* (ie. $\partial c = 0$) versus when a chain is a boundary of something (ie. $c = \partial c'$ for some $c' \in C_{k+1}(M)$). Geometrically, this measures holes in a space.



Trivial



Non-trivial

From Singular to de Rham

So far, we have developed two types of cohomology:

- De Rham $\longrightarrow$ Measures when closed forms are exact.
- Singular $\longrightarrow$ Measures the holes in a space.

We want to relate these two theories. The following will help with this:

- The singular k-simplex $\sigma : \Delta_k \to M$ induces the homomorphism $\sigma^* : \Omega^k(M) \to \Omega^k(\Delta_k)$, so $\sigma^*\omega$ is a k-form on Δ_k for $\omega \in \Omega^k(M)$.
- We can define integration of simplices as $\int_\sigma \omega = \int_{\Delta_k} \sigma^*\omega$. This can be extended to chains:
  - For the chain $c = \sum n_i \sigma_i$, we have that $\int_c \omega = \sum n_i \int_{\sigma_i} \omega$.

The Hom Ψ

This definition of integration of chains gives us the following:

Definition

- The homomorphism $\Psi(\omega) : C_k(M) \to \mathbb{R}$ is defined by $\Psi(\omega)(c) = \int_c \omega$.
- Ψ is also linear in ω. That is, given a k-form ω, the map $\Psi : \Omega^k(M) \to C^k(M; \mathbb{R})$ gives a homomorphism from $C_k(M)$ to $\mathbb{R}$ defined by $\Psi(\omega) = \Psi(\omega)(c)$. After some calculations utilizing Stokes' Theorem, we can see that $\Psi(d\omega)(\sigma) = \delta(\Psi(\omega))(\sigma)$ yielding the commutative diagram on the next slide.

The Hom Ψ

$$\begin{tikzcd}
\Omega^{k-1}(M) \arrow[r, "\Psi"] \arrow[d, "d"] & C^{k-1}(M;\mathbb{R}) \arrow[d, "\delta"] \\
\Omega^{k}(M) \arrow[r, "\Psi"] & C^{k}(M;\mathbb{R})
\end{tikzcd}$$

Figure 1

Ψ^* and the de Rham Theorem

Proposition

Figure 1 induces a homomorphism $\Psi^* : H^k_{\mathrm{dR}}(M) \to H^k(M;\mathbb{R})$.

The de Rham Theorem

For all differential manifolds M, the homomorphism $\Psi^* : H^k_{\mathrm{dR}}(M) \to H^k(M;\mathbb{R})$ is an isomorphism.

To conclude, we have the following corollary:

Corollary

We can entirely use calculus and differential geometry to determine the topology of a manifold. Conversely, we can examine the differential geometry of a manifold using only topology.

The End

Thank You!