A Diffeological Construction of Singer's Universal Connection

Dion Mann

Diffeology Seminar
9 June 2026

Review of Bundles and Connections

Principal Bundles

Let $\pi : P \to X$ a smooth surjection of manifolds and G a Lie group which acts freely and differentiably on P. We say $\pi : P \to G$ is a principal G-bundle if:

1. For each $p \in P$, the fiber $\pi^{-1}(p)$ is the G-orbit of p.
2. There is a family of G-equivariant local trivializations. That is, there is an open cover $X = \bigcup_\alpha U_\alpha$ and diffeomorphisms

   $$T_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$$

   satisfying $T_\alpha(x, gg') = T_\alpha(x, g)g'$.

Connections on Principal Bundles

Theorem

If $\pi : P \to X$ is a principal G-bundle, then there exists a connection $A \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$.

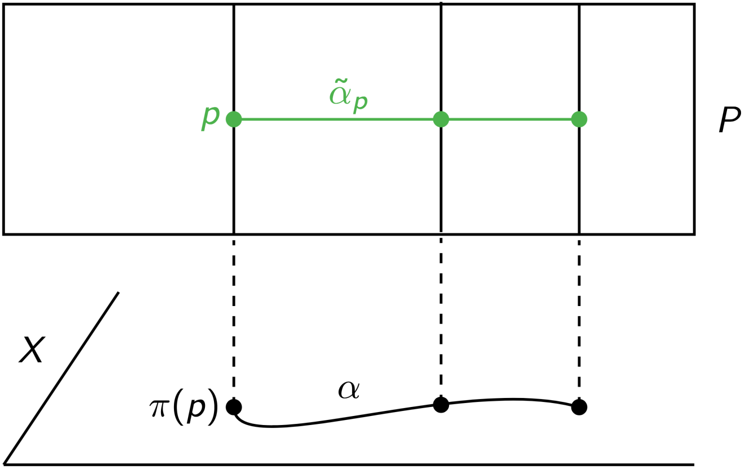
Every path α in the base X has a lift $\tilde{\alpha}$ into P such that

$$A(\tilde{\alpha}'(t)) = 0$$

for all t. In other words: $\tilde{\alpha}$ is horizontal.

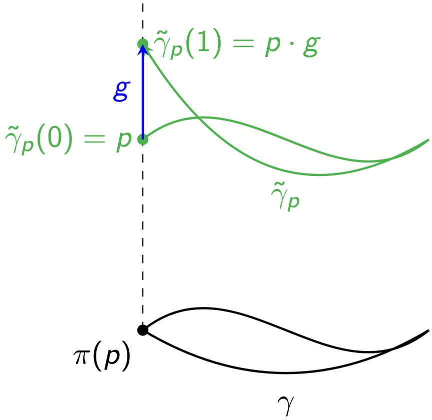
This means every principal bundle $\pi : P \to X$ admits a "horizontal lifting mechanism" which takes a path in X and gives a horizontal path in P.

Visual: Horizontal Lift



Horizontal Lifts of Loops

Let $\pi : P \to X$ be a principal G-bundle with a connection $A \in \Omega^1(P, \mathfrak{g})$. A loop γ in X may not horizontally lift to a loop in P:



In general: $\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot g$ for some (possibly nontrivial) $g \in G$.

Holonomy

Denote by $\mathcal{L}(X, x_0)$ the collection of loops in X based at x_0.

Definition: Holonomy Representation

For a connection A and a fixed point $\xi_0 \in \pi^{-1}(x_0)$, the holonomy representation of A at ξ_0 is the map

$$H^A_{\xi_0} : \mathcal{L}(X, x_0) \to G, \quad \tilde{\gamma}(1) = \xi_0 \cdot H^A_{\xi_0}(\gamma).$$

It can be shown that $H^A_{\xi_0}$ respects path-concatenation: given $\gamma \vee \sigma \in \mathcal{L}(X, x_0)$ (the path following σ then following γ), we have

$$H^A_{\xi_0}(\gamma \vee \sigma) = H^A_{\sigma(1)}(\gamma) \cdot H^A_{\xi_0}(\sigma).$$

Introduction: Singer's Universal Connection

Main Idea

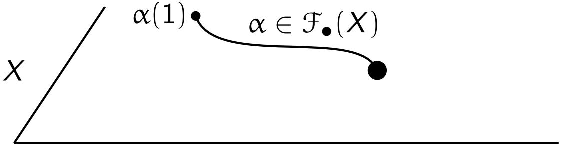
We begin with any pointed, path-connected smooth manifold $(X, \bullet)$.

Goal:

Construct a natural principal bundle-connection pair associated to $(X, \bullet)$.

Singer's Path Spaces

Let $\mathcal{F}_\bullet(X)$ be the set of all $\mathcal{C}^\infty$-paths $\alpha : [0,1] \to X$ with $\alpha(0) = \bullet$.



The "target map"

$$\tau : \mathcal{F}_\bullet(X) \to X, \qquad \tau(\alpha) = \alpha(1)$$

defines a natural projection.

Observations

- The fiber $\tau^{-1}(x)$ is the set of all paths starting at $\bullet$ and ending at x.
- The projection $\tau : \mathcal{F}_\bullet(X) \to X$ looks like a "principal $\mathcal{L}(X, \bullet)$-bundle."

Singer's Universal Connection

There is a natural notion of "connection" on $\tau : \mathcal{F}_\bullet(X) \to X$, which Singer calls the universal connection.

Goal:

Given $\alpha : [0,1] \to X$ and a "point" $\beta_0 \in \tau^{-1}(\alpha(0)) \subseteq \mathcal{F}_\bullet(X)$, we define the "horizontal lift" $\bar{\alpha}_{\beta_0} : [0,1] \to \mathcal{F}_\bullet(X)$ of α starting at β_0.

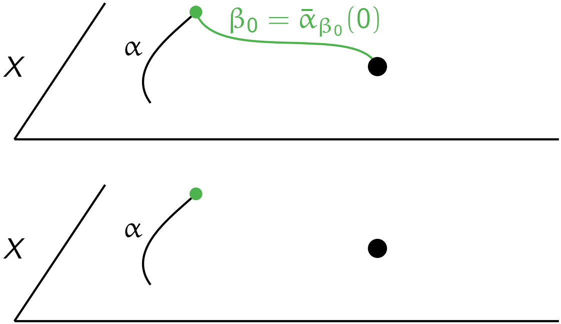
Since $\bar{\alpha}_{\beta_0}$ is a path of paths in X, we describe snapshots

$$\bar{\alpha}_{\beta_0}(s) : [0,1] \to X$$

for each $s \in [0,1]$.

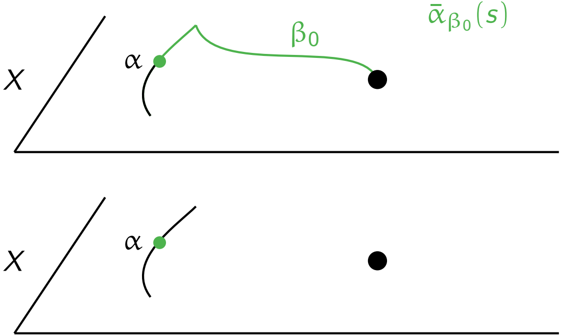
Singer's Universal Connection

For $s = 0$, we need $\bar{\alpha}_{\beta_0}(0) = \beta_0$.



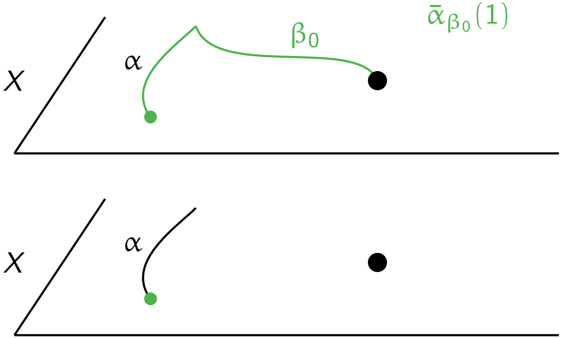
Singer's Universal Connection

For any intermediate $s \in (0,1)$:



Singer's Universal Connection

For $s = 1$:



Diffeologizing Singer's Construction

Goal:

We make this construction rigorous using the theory of diffeology.

We then discuss an application of the diffeological universal connection in proving a reconstruction theorem for bundles with connection.

Bundles, Connections, and Holonomy in Diffeology

Why Diffeology?

It seems that diffeology is a natural framework to study Singer's universal connection, e.g. to make sense of statements like:

- The projection $\tau : \mathcal{F}_\bullet(X) \to X$ is a principal bundle.
- The universal connection is a connection.

We will see later that diffeology is a natural arena for path spaces and path concatenation.

Diffeological Principal Bundles

Definition (Iglesias-Zemmour 2013): Diffeological Principal Bundles

Let G be a diffeological group which smoothly acts on a diffeological space P from the right. A smooth surjection $\pi : P \to X$ is a diffeological principal G-bundle if:

1. The map $P \times G \to P \times P, (p, g) \mapsto (p, pg)$ is a diffeomorphism onto its image.
2. The bundles $P \to P/G$ and $P \to X$ are equivalent.

The first item above implies that $P \to P/G$ is a diffeological bundle with fiber G (and that the G-action is free).

Remark

Diffeological G-bundles may be defined using families of G-equivariant local trivializations along each plot of X.

Towards Diffeological Connections: Path Spaces

Connections on diffeological principal bundles $\pi : P \to X$ are defined as a "horizontal projection" of paths in P. We need some notation.

Definition: Path Spaces

Let Y be a diffeological space and $\mathcal{D}$ its equipped diffeology.

1. The path space $\mathcal{P}Y \subseteq \mathcal{D}$ of Y is the set of all plots $\alpha : I \to Y$, where $I \subseteq \mathbb{R}$ is an open interval (may be $\mathbb{R}$).
2. The initialized path space is the set
$$\hat{\mathcal{P}}Y = \{(\alpha, t) \in \mathcal{P} \times \mathbb{R} : t \in \mathrm{dom}(\alpha)\}$$
3. The loop space $\mathcal{L}(Y, y_0)$ of Y based at y_0 is the set of all $\gamma \in \mathcal{P}Y$ such that $0, 1 \in \mathrm{dom}(\gamma)$ and $\gamma(0) = \gamma(1) = y_0$.

Note: all are equipped with the functional diffeology.

Diffeological Connections

Definition (Iglesias-Zemmour 2013): Diffeological Connection

A diffeological connection on a diffeological principal G-bundle $\pi : E \to X$ is a smooth map

$$A : \hat{\mathcal{P}}E \to \mathcal{P}E, \quad (\tilde{\alpha}, t) \mapsto A_t\tilde{\alpha}$$

satisfying a variety of conditions (e.g. $A_t(A_t\tilde{\alpha}) = A_t\tilde{\alpha}$, reparameterization invariance, G-equivariance, $\ldots$).

The path $A_t\tilde{\alpha}$ is the horizontal projection of α at time t.

Definition: Horizontal Path

A path $\tilde{\alpha} : I \to E$ is A-horizontal if $A_t\tilde{\alpha} = \tilde{\alpha}$ for some $t \in \mathrm{dom}(\tilde{\alpha})$.

Horizontal Lifts

Given a diffeological connection A on $\pi : E \to X$, we can horizontally lift paths in X to paths in E:

Definition/Theorem: Horizontal Lift

There is a map $\mathrm{hor}_A : \mathrm{ev}_X^*(E) \to \mathcal{P}E$, where

$$\mathrm{ev}_X^*(E) = \{(\alpha, t, \xi) \in \hat{\mathcal{P}}X \times E : \pi(\xi) = \alpha(t)\},$$

such that $\mathrm{hor}_A(\alpha, t, \xi)$ is horizontal. The path $\mathrm{hor}_A(\alpha, t, \xi)$ is called the horizontal lift of α starting at ξ.

Holonomy

Let $\pi : E \to X$ be a diffeological principal G-bundle with a connection A. The notion of holonomy is virtually the same as in the classical setting:

Definition: Holonomy Representation

For $\gamma \in \mathcal{L}(X, x_0)$ and $\xi_0 \in \pi^{-1}(x_0)$, define $H^A_{\xi_0}(\gamma) \in G$ by the condition

$$\mathrm{hor}_A(\gamma, 0, \xi_0)(1) = \xi_0 \cdot H^A_{\xi_0}(\gamma).$$

The smooth map $H^A_{\xi_0} : \mathcal{L}(X, x_0) \to G$ is called the holonomy representation of A at ξ_0.

We will see that, in some sense, we may view $H^A_{\xi_0} : \mathcal{L}(X, x_0) \to G$ as a smooth “group homomorphism” from the loop space $\mathcal{L}(X, x_0)$ to G.

Loop Group?

For a diffeological space Y, let $\alpha, \beta \in \mathcal{P}Y$ such that $1 \in \mathrm{dom}(\alpha)$, $0 \in \mathrm{dom}(\beta)$, and $\beta(0) = \alpha(1)$. We can define the path-concatenation $\beta \vee \alpha$ (read "β after α") by

$$(\beta \vee \alpha)(t) = \begin{cases} \alpha(2t) & \text{if } t \leqslant \frac{1}{2}, \\ \beta(2t-1) & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

Then $\vee$ defines a binary operation on $\mathcal{L}(X, x_0)$, but does not make $\mathcal{L}(X, x_0)$ into a group.

Holonomy Representation as a Group Homomorphism

There is an equivalence relation $\sim$ on $\mathcal{L}(X, x_0)$ which…

- …makes $\Omega_{x_0}(X) := \mathcal{L}(X, x_0)/\sim$ a diffeological group under $\vee$.
- …makes $H^A_{\xi_0} : \mathcal{L}(X, x_0) \to G$ descend to a smooth homomorphism

$$H^A_{\xi_0} : \Omega_{x_0}(X) \to G.$$

Definition (Gibilisko 1997): Retrace Equivalence

Define the following operations on $\mathcal{L}(X, x_0)$:

1. $\gamma \mapsto \gamma \circ f$ for $f \in \mathrm{Diff}([0,1])$ such that $f(0) = 0$ and $f(1) = 1$.
2. $\gamma \vee \sigma \mapsto \gamma \vee \eta \vee \eta^{\leftarrow} \vee \sigma$, where $\eta^{\leftarrow}(t) = \eta(1-t)$.

Then the equivalence relation $\sim$ finitely generated by operations (1) and (2) is called retrace equivalence. Put $\Omega_{x_0}(X) := \mathcal{L}(X, x_0)/\sim$ and denote by $[\gamma]$ a class of retrace-equivalent loops.

Note that retrace equivalence can be extended to arbitrary paths (not just loops): two paths α and β are retrace equivalent if α can be obtained from β by a finite sequence of the retrace operations.

Remark

There are other "group-like" relations on $\mathcal{L}(X, x_0)$, e.g. thin homotopy equivalence, to consider. We will only use retrace equivalence.

The Diffeological Universal Connection

Diffeological Path Spaces

Let $(X, \bullet)$ be a pointed, path-connected *diffeological space*.

This time we let $\mathcal{F}_\bullet(X)$ be the set of all *retrace-equivalent* smooth paths $\alpha : [0,1] \to X$ with $\alpha(0) = \bullet$ equipped with the natural diffeology. There is again the target map

$$\tau : \mathcal{F}_\bullet(X) \to X, \quad \tau([\alpha]) = \alpha(1).$$

Lemma

The loop space $\Omega_\bullet(X)$ acts smoothly on $\mathcal{F}_\bullet(X)$ on the right by path-concatenation, with the natural diffeologies.

Lemma

The projection $\tau : \mathcal{F}_\bullet(X) \to X$ is a diffeological principal $\Omega_\bullet(X)$-bundle.

Proof (outline).

- A plot $r \mapsto x_r$ for X determines a smooth family of "radial paths", i.e. for a fixed point r_0 and any r, there is a path α_r from x_{r_0} to x_r which varies smoothly in r.
- The map $\Phi : X \to \mathcal{F}_\bullet(X)/\Omega_\bullet(X)$ sending x to the class of any path from $\bullet$ to x is smooth, since we can write locally

$$\Phi(x_r) = [\alpha_r \vee \lambda_{x_{r_0}}] \vee \Omega_\bullet(X), \quad \text{where } \lambda_{x_{r_0}} \text{ is any path from } \bullet \text{ to } x_{r_0}.$$

- Φ is a diffeomorphism, since $\Phi^{-1}([\alpha] \vee \Omega_\bullet(X)) = \alpha(1)$.

☐

Diffeological Universal Connection

Recall a diffeological connection is a "horizontal projector" of paths into the *total space*. A modification of Singer's universal connection is possible.

Notation

- For any map $\beta : I \to X$ and times $r, s \in I$, we write $\beta^{r \to s} : [0,1] \to X$ to be the path following β from $\beta(r)$ to $\beta(s)$ in unit time. More precisely:

  $$\beta^{r \to s}(t) = \beta((1-t)r + ts).$$

- We use brackets $[\tilde{\alpha}] : I \to \mathcal{F}_\bullet(X)$ to denote a path in $\mathcal{F}_\bullet(X)$: for each $s \in I$, evaluating $s \mapsto [\tilde{\alpha}(s)]$ gives a retrace-equivalence class of paths

  $$\tilde{\alpha}(s) : [0,1] \to X$$

  beginning at $\bullet$.

Theorem

For a path $[\tilde{\alpha}] : I \to \mathcal{F}_\bullet(X)$ and an initial time $t_0 \in I$, define a path $\Theta_{t_0}[\tilde{\alpha}] : I \to \mathcal{F}_\bullet(X)$ by

$$\Theta_{t_0}[\tilde{\alpha}](s) = [(\tau \circ [\tilde{\alpha}])^{t_0 \to s}] \vee [\tilde{\alpha}(t_0)].$$

Then the map

$$\Theta : \hat{\mathcal{P}}\mathcal{F}_\bullet(X) \to \mathcal{P}\mathcal{F}_\bullet(X), \quad ([\tilde{\alpha}], t) \mapsto \Theta_t[\tilde{\alpha}]$$

defines a diffeological connection on $\tau : \mathcal{F}_\bullet(X) \to X$.

Relationship to Singer's Connection

The diffeological universal connection gives rise to a horizontal lifting function, which is the connection that Singer described:

Proposition

Given $\alpha : (a, b) \to X$, an initial time $t_0 \in (a, b)$, and an initial "point" $\beta_0 \in \tau^{-1}(\alpha(t_0))$, the horizontal lift

$$\mathrm{hor}_\Theta(\alpha, t_0, \beta_0) : (a, b) \to \mathcal{F}_\bullet(X)$$

is given by

$$\mathrm{hor}_\Theta(\alpha, t_0, \beta_0)(s) = [\alpha^{t_0 \to s} \vee \beta_0].$$

In other words: Singer's lifts are horizontal with respect to the diffeological universal connection.

Bundle-Connection Reconstruction and Holonomy

Bundle-Connection Reconstruction

We can use the diffeological universal connection to prove a *reconstruction-style* theorem. The original idea seems to be due to S. Kobayashi in the 50s:

Theorem (Kobayashi 1954)

Let $(M, \bullet)$ be a pointed, path-connected smooth manifold and G a Lie group. Given a continuous homomorphism $H : \Omega_\bullet(M) \to G$, there exists a principal G-bundle $\pi : (E, \xi_0) \to (M, \bullet)$ with a connection A such that $H^A_{\xi_0} = H$.

Variants of the Reconstruction Theorem

Overtime, Kobayashi's reconstruction theorem has been formulated in various ways, including but not limited to:

- (Barrett 1991) Reconstruction of bundles over manifolds in the context of Yang-Mills theory, where the loop space is obtained by quotienting via thin-homotopy equivalence.
- (Caetano and Picken 1994) Axiomatic definition of holonomy and reconstruction of bundles over manifolds.
- (Collier, Lerman, Wolbert 2016) Reconstruction theorem for bundles over stacks.

We show a similar result holds for diffeological bundlles with connection using category-theoretic language.

The Holonomy Category

Fix a diffeological group G. We introduce a category Hol^G:

- Objects are triples (X, x_0, H) where (X, x_0) is a pointed, path-connected diffeological space and $H : \Omega_{x_0}(X) \to G$ is a smooth homomorphism.

- Morphisms are arrows $f : (X, x_0, H) \to (X', x_0', H')$ such that $f : (X, x_0) \to (X', x_0')$ is a pointed smooth map and

  $$H'([f \circ \gamma]) = H([\gamma]),$$

  for all $[\gamma] \in \Omega_{x_0}(X)$.

The Category of Bundle-Connection Pairs

Fix a diffeological group G. We introduce a category $\mathrm{B}_\bullet\mathcal{A}^G$:

- Objects are pairs $(\pi : (E, \xi_0) \to (X, x_0), A)$ where $\pi : (E, \xi_0) \to (X, x_0)$ is a pointed, diffeological principal G-bundle and A is a connection on π.
- Morphisms are pairs

  $$(F, f) : (\pi, A) \to (\pi', A')$$

  which is a G-bundle morphism from π to π' and also satisfies

  $$F \circ A_t\tilde{\alpha} = A'_t(F \circ \tilde{\alpha}).$$

Morphisms in $\mathrm{B}_\bullet\mathcal{A}^G$

More visually, a morphism

$$(F, f) : (\pi : (E, \xi_0) \to (X, x_0), A) \to (\pi' : (E', \xi_0') \to (X', x_0'), A')$$

in $\mathrm{B}_\bullet\mathcal{A}^G$ satisfies the following two commutative diagrams:

$$\begin{tikzcd}
(E, \xi_0) \arrow[r, "F"] \arrow[d, "\pi"'] & (E', \xi_0') \arrow[d, "\pi'"] \\
(X, x_0) \arrow[r, "f"'] & (X', x_0')
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{\mathcal{P}}E \arrow[r, "F_* \times \mathrm{id}"] \arrow[d, "A"'] & \hat{\mathcal{P}}E' \arrow[d, "A'"] \\
\mathcal{P}E \arrow[r, "F_*"'] & \mathcal{P}E'
\end{tikzcd}$$

where F_* is the map sending $\tilde{\alpha} : I \to E$ to $F \circ \tilde{\alpha} : I \to E'$.

The Forgetful-Holonomy Functor

There is a natural "forgetful" functor $\mathrm{B}_\bullet\mathcal{A}^G \to \mathrm{Hol}^G$ as follows:

- Given a bundle-connection pair $(\pi : (E, \xi_0) \to (X, x_0), A)$, we obtain the holonomy representation $H^A_{\xi_0} : \Omega_{x_0}(X) \to G$, and hence an object $(X, x_0, H^A_{\xi_0})$ of Hol^G.

- Given a morphism $(F, f) : (\pi, A) \to (\pi', A')$, one shows that $H^{A'}_{\xi'_0}([f \circ \gamma]) = H^A_{\xi_0}([\gamma])$ for all $[\gamma] \in \Omega_{x_0}(X)$, so f defines a morphism in Hol^G.

The Reconstruction Functor

Theorem

There is a “reconstruction functor” $\mathsf{Hol}^G \to \mathsf{B}_\bullet\mathcal{A}^G$ which is inverse to the forgetful functor $\mathsf{B}_\bullet\mathcal{A}^G \to \mathsf{Hol}^G$, i.e. there is an equivalence of categories

$$\mathsf{Hol}^G \leftrightarrows \mathsf{B}_\bullet\mathcal{A}^G.$$

In particular, we have a Kobayashi-style reconstruction theorem for diffeological bundles.

Bundle Reconstruction

Let $(X, \bullet, H)$ be any Hol^G-object. We have the principal $\Omega_\bullet(X)$-bundle $\tau : \mathcal{F}_\bullet(X) \to X$.

We can use $H : \Omega_\bullet(X) \to G$ to amalgamate $\tau : \mathcal{F}_\bullet(X) \to X$ from $\Omega_\bullet(X)$ to G and obtain a diffeological principal G-bundle $\tau_H : P_H \to X$, where

$$P_H = \mathcal{F}_\bullet(X) \times_H G$$

is the associated bundle of $\tau : \mathcal{F}_\bullet(X) \to X$ via H.

Elements of P_H are equivalence classes $[\![[\alpha], g]\!]_H$ where

$$([\alpha], g) = ([\alpha] \vee [\gamma], H([\gamma])^{-1} g), \quad \text{for all } [\gamma] \in \Omega_\bullet(X).$$

Bundle-Connection Reconstruction

The universal connection Θ on $\tau : \mathcal{F}_\bullet(X) \to X$ induces a map $\Theta^H : \hat{\mathcal{P}}P_H \to \mathcal{P}P_H$ by the formula

$$\Theta^H_{t_0}(\llbracket [\tilde{\alpha}], g \rrbracket_H) = \llbracket \Theta_{t_0}[\tilde{\alpha}], g \rrbracket_H .$$

Lemma

1. $\tau_H : P_H \to X$ is a diffeological principal G-bundle.
2. Θ^H is a diffeological connection on $\tau_H : P_H \to X$.

So we obtain the $\mathrm{B}_\bullet\mathcal{A}^G$-object $(\tau_H : P_H \to X, \Theta^H)$.

The Reconstruction on Morphisms

The assignment $(X, \bullet, H) \mapsto (\tau_H : P_H \to X, \Theta^H)$ extends to a functor $\mathrm{Hol}^G \to \mathrm{B}_\bullet\mathcal{A}^G$ as follows:

Given a Hol^G-morphism $f : (X, \bullet, H) \to (X', \bullet', H')$, we define a map $F_f : P_H \to P_{H'}$ by

$$F_f(\llbracket [\alpha], g \rrbracket_H) = \llbracket [f \circ \alpha], g \rrbracket_{H'}.$$

One can show that F_f is well-defined and satisfies the morphism condition in $\mathrm{B}_\bullet\mathcal{A}^G$, hence (F_f, f) is a morphism in $\mathrm{B}_\bullet\mathcal{A}^G$.