An Invitation to Mathematical Physics

Dion Mann

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Why Mathematical Physics?

Mathematics in Physics

There is lots of mathematics in physics!

Examples:

You cannot do...

- Electromagnetism without vector calculus.
- Quantum Mechanics without linear algebra/functional analysis.
- Condensed Matter without topology.
- ...And so on.

But, maybe it is suprising that...

Physics in Mathematics

...there is physics in mathematics, too!

Classifying Simple Groups

As an example, let's investigate an interesting connection between algebra and complex analysis.

Simple Groups

A finite group $G \neq \{e\}$ is called simple if it has no normal subgroup except for $\{e\}$ and G.

Sporadic Simple Groups

In the classification of finite simple groups, some groups fit into an infinite family, like $\mathbb{Z}/p$ for a prime p or the alternating groups A_n for $n \geq 5$, and 26 of them don't: they are called sporadic finite simple groups.

The Monster Group

The monster group $\mathbb{M}$ is the largest sporadic finite simple group, with

$$|\mathbb{M}| \approx 8 \times 10^{53}.$$

The j-function

Let $\mathcal{H} \subseteq \mathbb{C}$ denote the complex upper-half plane; that is,

$$\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

Definition: j-function

There is a holomorphic function $j : \mathcal{H} \to \mathbb{C}$ such that:

1. for $\tau_1, \tau_2 \in \mathcal{H}$, we have $j(\tau_1) = j(\tau_2)$ if and only if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d} \qquad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

2. j has a pole at $i\infty$ with residue 1

The function as described above is unique (up to an additive constant) and is called the j-function.

Series Expansion for the *j*-function

The j-function has a series expansion

$$j(q) = \frac{1}{q} + 744 + 196\ 884q + 12\ 493\ 760q^2 + 864\ 299\ 970q^3 + \cdots ,$$

where $q = e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$.

It turns out that j and $\mathbb{M}$ are related!

Group Representations

How the monster $\mathbb{M}$ and the j-function are related has to do with group representations.

Definition: Group Representation

An n-dimensional representation of a group G is a group homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

In other words, a representation of G is an assignment of a matrix to each $g \in G$. Recall that $(n \times n)$-matrices act on the vector space $\mathbb{C}^n$ by matrix multiplication.

Monstrous Moonshine

Recall $j(q) = \frac{1}{q} + 744 + 196\,884q + 12\,493\,760q^2 + 864\,299\,970q^3 + \cdots$

The First Surprise

The first surprise is that $196\,884 = 196\,883 + 1$.

The numbers 1 and 196 883 are the dimensions of the smallest (irreducible) representations of $\mathbb{M}$.

John Conway and Simon Norton observed (1979) that this pattern continues:

$$12\,493\,760 = 21\,296\,876 + 196\,883 + 1$$

is also a sum of representation dimensions.

Monstrous Moonshine

This unexpected connection between $\mathbb{M}$ and the j-function, going beyond coefficients of q and q^2, is called monstrous moonshine.

Borchers' Proof for Monstrous Moonshine

Richard Borcherds received the Fields Medal in 1998 for proving monstrous moonshine.

The proof? It used...

The No-Ghost Theorem

A theorem describing some properties of a certain quantization functor for bosonic strings. This is the domain of string theory!

Borcherds' proof also depended on a construction due to Igor Frenkel, James Lepowsky, and Arne Meurman. The construction was a physical model of a bosonic string moving on a 24-dimensional torus.

So Why Mathematical Physics?

In monstrous moonshine, we saw how a surprising mathematical result was proved using math inspired by physical theory.

Many problems in mathematical physics are problems in mathematics, but with a significant context (which varies from problem to problem) in physics.

Projects in Mathematical Physics

Mathematical Foundations for Physics

Can we study physics as a mathematical discipline?

To do so, we need a set of axioms—statements which we choose to accept as true.

All areas of mathematics have results which can be derived starting from a fixed set of axioms.

Euclid's *Elements*

Euclidean geometry is an example of an axiomatic system:

Some Axioms of Geometry

All theorems in Euclidean geometry can be proved starting from axioms like (translated from Euclid's *Elements*)...

- "To draw a straight line from any point to any point," or
- "To describe a circle with any center and radius."

Some observations:

1. When physically interpreting Euclid's axioms, the statements were considered "obviously true."
2. Theorems of Euclidean geometry are valid independently of their meaning in the physical world.

Classical Probability

Probability was initially described intuitively as follows:

Classical Probability

"The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible." –Pierre-Simon Laplace (1812).

Statements could be made and argued for in classical probability:

- **Ex**: A "fair" coin toss has a 50% chance of landing on heads.

But the mathematical framework was nowhere near on the level of e.g. algebra or geometry at the time.

Measure-Theoretic Probability

In the 1930s, mathematician Andrey Kolmogorov introduced an axiomatic system for probability.

Kolmogorov axioms

Given a measure space $(\Omega, \mathcal{F}, p)$, we assume:

1. $p(E) \geq 0$ for all $E \in \mathcal{F}$.
2. $p(\Omega) = 1$.
3. $p\Big(\bigcup_{n\geq 1} E_n\Big) = \sum_{n\geq 1} p(E_n)$ for any sequence $(E_n)_{n\geq 1}$ of disjoint sets in $\mathcal{F}$.

These are assumed to be "obviously true" under an intepretation of probability, but do not assume any interpretation to begin with.

Theorems of "probability theory" are valid independently of the physical world, but recover the statements of "classical probability."

Birth of a Mathematical Theory

In these examples, we see a mathematical theory historically follows the path:

Intuition $\rightarrow$ Axioms $\rightarrow$ Theorems and Interpretation/Application

Axiomatization of Physics?

Right now, physics is in the “intuition” stage. Can we, in the same way as geometry and probability, develop a system of axioms for physics?

Project: Hilbert’s 6th Problem (1900)

Also known as the Axiomatization of Physics problem. In Hilbert’s words:

“To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part.”

A set of axioms, if it exists, should...

- ...be “obviously true” when interpreted in the physical world.
- ...be enough to recover physical claims as mathematical theorems.

Axiomatizing Physics Leading to More Mathematics

It is important to note that individual areas of physics have been axiomatized or, at least, have a proposed set of axioms.

- General relativity (GR) uses the axioms of differential geometry.
- Quantum field theory (QFT) uses, for example, the Wightman axioms.

But Hilbert's sixth problem remains open—there is, to date, no single set of axioms describing both QFT and GR.

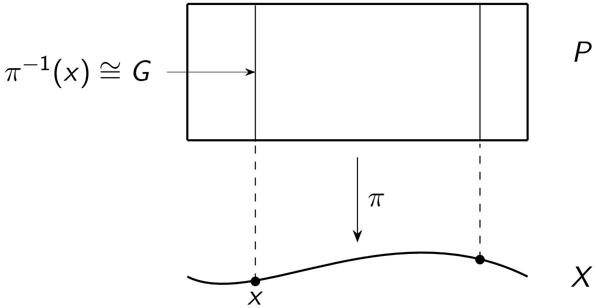
- The Wightman axioms assume spacetime is a fixed, "flat" space, but GR describes spacetime as being curved, for example.

Principal Bundles

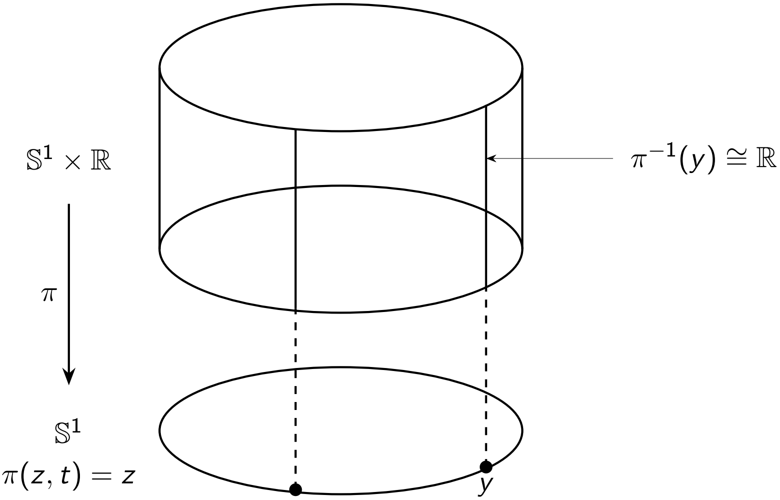
Let's look more closely at geometry: principal bundles.

Definition: Principal G-Bundles

A principal G-bundle is a surjective map $\pi : P \to X$ of spaces, where P (locally) looks like $X \times G$ for a group G. Each $\pi^{-1}(x) \subseteq P$ is called a fiber over x and is an isomorphic copy of G. Furthermore, G acts transitively on each fiber.

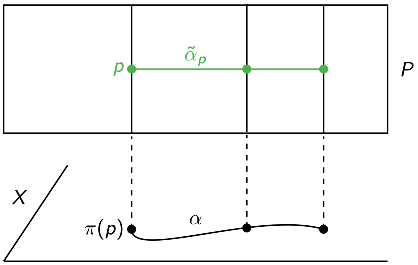


Example: Cylinder is a principal $\mathbb{R}$-bundle over the Circle



Connections on Principal Bundles

It turns out that on principal bundles $\pi : P \to X$, there is a way to lift paths on the base X into P "horizontally."



This is called a connection, because it connects the fibers together.

Mathematical Gauge Theory

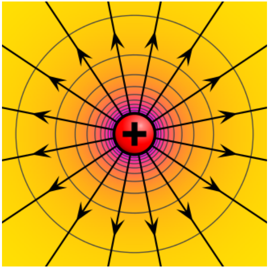
One can imagine that if the space P is distorted, then the horizontal lift would look more complicated. This is measured using curvature.

Gauge Theory

The study of connections and their curvature on principal bundles is called gauge theory. Gauge theory has led to significant results in mathematics, such as calculating topological invariants of bundles and generalizing the Gauss–Bonnet theorem (which connects global topology with local geometry).

Scalar Potentials

At the same time (around 1930s) mathematical gauge theory was being developed, physicists were studying symmetries of fundamental forces.



An electric field, for example, is represented by a vector field in $\mathbb{R}^3$.

Scalar Potentials

Recall that a conservative vector field $\vec{E}$ can be written as ∇V for some scalar function V. Such a V is called a scalar potential for $\vec{E}$.

Gauge Freedom in Electric Field

Note that if $\vec{E} = \nabla V$, then $\vec{E} = \nabla(V + c)$ for any constant c too. Conversely, any two choices of V making $\vec{E} = \nabla V$ differ by a constant.

Proof.

Suppose V_1 and V_2 satisfy $\nabla V_1 = \vec{E} = \nabla V_2$. Define $V = V_2 - V_1$. Then $\nabla V = 0$, so V is constant. □

In the context of the electric field $\vec{E} = \nabla V$, the scalar function V is called the electric potential. Any two electric potentials (for the same electric field) differ by a constant—this freedom is what physicists called gauge freedom.

Gauge Freedom in Electromagnetism

Gauge freedom occurs in the magnetic field $\vec{B}$ too. In this case, one has a vector potential $\vec{A}$, which means

$$\vec{B} = \nabla \times \vec{A}.$$

Claim:

$\nabla \times \vec{A}_1 = \nabla \times \vec{A}_2$ if and only if $\vec{A}_2 = \vec{A}_1 + \nabla\phi$ for some scalar function ϕ.

Gauge Freedom for Fundamental Forces

It turns out that all fundamental forces (electromagnetic force, weak/strong nuclear forces, and the gravitational force) share similar gauge freedoms/invariances.

It wasn't until the 1970s when the realization set in that

- Mathematicians studying connections on principal bundles, and
- Physicists studying gauge invariance of forces

were really studying the same object!

Gauge Theory

This is the main topic of gauge theory: connections on principal bundles.

Forces as Curvature of a Connection

Similar to how gravity (in general relativity) is described with curvature, each fundamental force of physics is described as a curvature of a connection on a principal G-bundle:

1. Electromagnetic force: uses the group $G = U(1) = \{z \in \mathbb{C} : |z| = 1\}$.
2. Weak nuclear force: uses the group

$$G = SU(2) = \{A \in \mathrm{Mat}_2(\mathbb{C}) : A\bar{A}^t = \mathrm{Id}_2,\ \det(A) = 1\}.$$

3. Strong nuclear force: uses the group $G = SU(3)$.

The gauge freedom in physics is a result of how the curvature of a connection changes from one “chart” (region in spacetime) to another.

Connections to String Theory

Today, many results predicted by the standard model of particle physics can be recovered as mathematical statements in gauge theory.

One attempt for a theory of everything is called string theory, which replaces particles with strings. Can we do gauge theory for strings?

Gauge Theory of Strings

If X is spacetime, we can view particles as points in X. Then strings can be viewed as paths $[0,1] \to X$ in X.

A gauge theory for strings would replace X with the space of paths $P(X)$.

Open Project

Given a (smooth) space X, is there a generalization of gauge theory to bundles of the form $\pi : E \to P(X)$, where $P(X)$ is the space of paths $[0,1] \to X$?

This project has been called higher gauge theory and is an active area of current research.

Quantization

A natural extension of gauge theory is to study quantization—a recipe for turning a classical system into a quantum analogue.

The first, precise appearence of "quantization" is due to the physicist Paul Dirac (1926):

Dirac's Quantization

A quantization is a map Q that sends real-valued functions f into (Hermitian) operators $Q(f) = \hat{f}$ of functions such that...

1. Q is linear.
2. Q respects the position and momentum operators from physics.
3. Q preserves certain brackets: $Q(\{f, g\}) = [\hat{f}, \hat{g}]$.

The No-Go Theorem

It turns out there is no such map Q on all classical observables satisfying all of Dirac's axioms. This result is Groenewold's Theorem (1946).

Fortunately, Groenewold's theorem has led to much interesting mathematics: one example we'll look at is deformation quantization.

Deformation of Algebras

Algebras

An algebra A over a field k (like $\mathbb{R}$ or $\mathbb{C}$) is a k-vector space that is also a ring. That means there is a multiplication operation $\mu_0 : A \times A \to A$, $\mu_0(x, y) = xy$, where $c(xy) = (cx)y = x(cy)$ for all scalars $c \in k$ and $x, y \in A$.

We can deform the algebraic product μ_0 into a new product μ by, for example, adding "correction terms" in a formal power series

$$\mu(x, y) = xy + \mu_1(x, y)\lambda + \mu_2(x, y)\lambda^2 + \cdots$$

where λ is a formal parameter and $\mu_i : A \times A \to A$ are (bilinear) maps.

Remark

The formal parameter λ is sometimes taken to be Planck's constant $\hbar$ or some multiple of it (like $i\hbar/2$). Note that $\lambda \to 0$ iff $\hbar \to 0$, in this case.

Gerstenhaber's Deformation Theory

The deformed product

$$\mu = \sum_{n \geq 0} \lambda^n \mu_n$$

can be required to satisfy certain conditions, like associativity:

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z).$$

Deformation Theory of Algebras

The study of deformed products on algebras is the subject of deformation theory, which is due to Murray Gerstenhaber in the 1960s.

Example of a Deformed Product

If we take $A = \mathbb{R}[x]$, we have the usual product of polynomials

$$\mu_0(f(x), g(x)) = f(x) \cdot g(x).$$

One way to deform μ_0 is to define $\mu_1(f(x), g(x)) = f'(x)g(x)$ and $\mu_n = 0$ for all $n \geq 2$. Then:

$$\mu(f(x), g(x)) = f(x)g(x) + f'(x)g(x)\lambda$$

is a deformed product.

One can check that μ is associative. It is also noncommutative, since

$$\mu(f(x), g(x)) = \mu(g(x), f(x)) + \big(f'(x)g(x) - f(x)g'(x)\big)\lambda$$

Deformation Quantization

Recall Dirac's quantization is a map Q that, in particular, needs to satisfy $Q(\{f,g\}) = [Q(f), Q(g)]$. By Groenewold's theorem, this is not possible for all f and g.

What if we relax the bracket condition?

Deformation Quantization

A deformation quantization of an algebra A is a deformation

$$\mu(f,g) = fg + \mu_1(f,g)\lambda + \mu_2(f,g)\lambda^2 \cdots$$

of A such that $\mu(f,g) - \mu(g,f) \to \{f,g\}$ as $\lambda \to 0$.

Such μ can actually exist! The no-go theorem of Groenewold is avoided.

Existence of Deformation Quantizations

When does a deformation quantization exist?

Existence of Deformation Quantizations

In 1998, Maxim Kontsevich got a Fields medal for proving that μ always exists for algebras of the form $A = \mathcal{C}^{\infty}(M)$, where M is a Poisson manifold.

Deformation Quantization for Fields?

Although Kontsevich proved a deformation quantization μ exists for a large class of algebras $A = \mathcal{C}^\infty(M)$, it is important to note these spaces are finite-dimensional. The fact that M is smooth is also used.

Open Project

Does a deformation quantization exist for spaces of fields? In physics, a field is a map $X \to V$ where X is spacetime and V is a vector space.

The algebra to deform is much more complicated. Its elements consist of functionals (functions of functions).

Final Remarks on Mathematical Physics

The projects discussed today—axiomatization of physics, gauge theory for strings, and deformation quantization for fields—are really projects in mathematics, but all can be interpreted in the context of physics.

Mathematical physics is a natural arena in which many areas of math are combined together.

This makes mathematical physics a great opportunity for mathematicians to come together and collaborate. Just as string theory explained a relation between the j-function and the Monster group, modern physics could reveal more mathematical connnections.

Where to see More

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