Diffeology

A Smooth Analogue of Topology

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Introduction

What is a Smooth Function?

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function $(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$.

Classical Smoothness

The function $f : \mathbb{R}^n \to \mathbb{R}$ is called smooth if it has *all* partial derivatives

$$\frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_n}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \dots$$

Examples of Smooth Functions

All polynomials (including the constants), like $f(x, y) = x^2 + xy + y^2$ and trigonometric functions, like $f(x, y, z) = \cos(x) \sin(yz)$, are smooth.

An Equivalent Characterization of Smoothness

A path is a function γ with $\operatorname{dom}(\gamma) \subseteq \mathbb{R}$ an open interval, which could be all of $\mathbb{R}$.

A sophisticated result due to J. Boman says we can characterize the smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ by *smooth paths* $\gamma : \mathbb{R} \to \mathbb{R}^n$:

Theorem (Boman 1967)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then f is smooth if and only if $f \circ \gamma$ is smooth for *every* smooth path $\gamma : \mathbb{R} \to \mathbb{R}^n$.

Boman's theorem generalizes to functions $f : \mathbb{R}^n \to \mathbb{R}^m$ by considering each component function. In particular:

Another Viewpoint of Smoothness

A smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ are precisely the ones that take smooth paths in $\mathbb{R}^n$ to smooth paths in $\mathbb{R}^m$.

Abstract Smoothness

In topology, we are interested in the continuous maps $f : X \to Y$ between abstract spaces. If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, so that

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

then the abstract definition of continuity recovers the usual definition.

Question:

In a similar fashion, can we study the abstract smooth maps $f : X \to Y$ between abstract spaces, in such a way that if $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then we recover the usual definition of smoothness?

Diffeology: Abstract Smoothness

The answer is: *yes!* We use the theory of diffeology, introduced in the 80s by the mathematician J.M Souriau and developed further by his student P. Iglesias-Zemmour.

Let $f : X \to Y$ be a map between two *sets* (thinking *abstract spaces*). We want smooth structures on X and Y to make sense of the statement

"γ is a smooth path in X."

Considering Boman's theorem, we can declare:

$$f : X \to Y \text{ is smooth} \iff f \circ \gamma \text{ is a smooth path in } Y \text{ for all smooth paths } \gamma \text{ in } X.$$

Axioms for a Diffeology

The idea is to consider maps (like paths) into X, and write down axioms that reflect properties of smooth maps.

Definition: Diffeology

Let X be any set. A diffeology on X is a collection $\mathcal{D}$ of maps $\psi : U \to X$, where $U \subseteq \mathbb{R}^n$ is any open subset (n is allowed to vary), such that:

1. If $n = 0$ (so $\psi : \mathbb{R}^0 \to X$ is a constant map), then $\psi \in \mathcal{D}$.
2. If $\psi : U \to X$ is in $\mathcal{D}$ and $f : V \to U$ is a (classically) smooth function from $V \subseteq \mathbb{R}^k$ to $U \subseteq \mathbb{R}^n$, then $\psi \circ f : V \to X$ is in $\mathcal{D}$.
3. If $\psi : U \to X$ is a map and there is a collection of open subsets U_α such that $\psi \restriction U_\alpha : U_\alpha \to X$ are all members of $\mathcal{D}$, then $\psi : U \to X$ is in $\mathcal{D}$.

We call elements of $\mathcal{D}$ plots for X and think of them as "smooth parameterizations of the set X."

Example: Standard Diffeology on $\mathbb{R}^n$

Consider $X = \mathbb{R}^n$. The standard diffeology of $\mathbb{R}^n$ is given by:

$$\psi : \mathbb{R}^k \supseteq U \to \mathbb{R}^n \text{ is a plot} \iff \psi : U \to \mathbb{R}^n \text{ is classically smooth.}$$

Let us check the standard diffeology really is a diffeology:

1. If ψ is constant, then ψ is smooth, so ψ is a plot.
2. Let $\psi : U \to \mathbb{R}^n$ be a plot and $f : V \to U$ be a classically smooth function from the subset $V \subseteq \mathbb{R}^k$. Then $\psi : U \to \mathbb{R}^n$ is classically smooth, hence $\psi \circ f : V \to \mathbb{R}^n$ is too (by the chain rule).
3. The last axiom follows because differentiability is a local property!

Indeed, the very axioms of diffeology are taken from properties of "smooth parameterizations" of $\mathbb{R}^n$.

Example: Spaghetti Diffeology on $\mathbb{R}^2$

We could have chosen a different diffeology to get a different smooth structure. For example,

Spaghetti Diffeology

Consider the following diffeology on $\mathbb{R}^2$: a map $\psi : U \to \mathbb{R}^2$ is a plot if and only if ψ factors through a smooth map into $\mathbb{R}$. More precisely, there are smooth maps q and F making the diagram commute:

$$\begin{tikzcd}
\mathbb{R}^n \supseteq U \arrow[rr, "\psi"] \arrow[dr, "q"'] & & \mathbb{R}^2 \\
& \mathbb{R} \supseteq V \arrow[ur, "F"'] &
\end{tikzcd}$$

Note every spaghetti plot is a standard plot, but not conversely:

- $f(x, y) = (x, y)$ can't be expressed as a smooth map $\mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}^2$

Example: Trivial Diffeologies

Every set X has at least two diffeologies:

Trivial/Coarse Diffeology

The trivial/coarse diffeology consists of *all* maps $U \to X$ for all open subsets $U \subseteq \mathbb{R}^n$ and all $n \geq 0$. This is the largest diffeology on X—everything is a smooth parameterization!

Discrete Diffeology

The discrete diffeology consists of *only locally constant maps* $U \to X$. This is the smallest diffeology on X.

The technical reason for *locally* constant is because of the 3rd-axiom for a diffeology.

Diffeological Spaces and Smooth Maps

A set X equipped with a diffeology $\mathcal{D}$ is called a diffeological space. Since $\mathcal{D}$ gives X a smooth structure, we understand the statement

“Let γ be a smooth path in X”

as “let $\gamma : U \to X$ be a plot for X, where $U \subseteq \mathbb{R}$” is open.

Recall Boman’s theorem says that smooth maps should take smooth paths to smooth paths. Therefore, we make the definition:

Definition: Diffeologically Smooth

A map $f : X \to Y$ between diffeological spaces is said to be (diffeologically) smooth if $f \circ \psi$ is a plot for Y for each plot ψ for X.

If $f : X \to Y$ has a smooth inverse, we say f is a diffeomorphism.

Example: Diffeologically Smooth Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ be equipped with the standard diffeologies.
Then Boman's theorem can be restated as:

Theorem (Boman 1967)

A map $f : X \to Y$ is classically smooth if and only if it is diffeologically smooth, where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ have the standard diffeologies.

Constructions in Diffeology

Basic Constructions in Diffeology

Given some diffeological spaces, say X and Y, we can construct new diffeological spaces.

Main Idea: Key Maps

Every "construction" comes with key maps. For example, a product space $X \times Y$ comes with projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. The natural diffeology is the one that makes π_1 and π_2 smooth.

Remark:

Generally, there are many diffeologies that will accomplish this. The most interesting one is always either the "largest" or "smallest" one—whichever is the nontrivial one!

Construction: Product Spaces

Let X and Y be diffeological spaces.

Definition: Product Diffeological Space

The product space is the set $X \times Y$ with plots

$$\psi : U \to X \times Y, \qquad \psi(r) = (\psi_1(r), \psi_2(r))$$

whose components $\psi_1 : U \to X$ and $\psi_2 : U \to Y$ are plots for X and Y.

This product diffeology is the "largest" one making the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ smooth.

Construction: Subspaces

Let X be a diffeological space and $A \subseteq X$ any subset.

Definition: Subspace Diffeology

The subspace is the subset $A \subseteq X$ with plots

$$\psi : U \to A$$

such that, when viewed as a map $U \to X$, is a plot for X.

This subspace diffeology is the "largest" one making the inclusion $i : A \hookrightarrow X$ smooth.

Example: Diffeology on the Rationals

Consider the rationals $\mathbb{Q} \subseteq \mathbb{R}$, where $\mathbb{R}$ has the standard diffeology. The subspace diffeology on $\mathbb{Q}$ is discrete—the only plots for $\mathbb{Q}$ are the (locally) constant ones.

Proof.

A plot $f : U \to \mathbb{Q}$ is, when viewed as a map into $\mathbb{R}$, smooth. In particular, f is a continuous map into the real line. By the intermediate value theorem, f is constant on each connected component of U. □

The fact that $\mathbb{Q}$ is "discrete" seems to match our intuition. Note that $\mathbb{Q}$ is *not* discrete topologically (open subsets look like intervals with holes).

Construction: Quotient Spaces

Let $\sim$ be an equivalence relation on the diffeological space X.

Definition: Quotient Diffeology

The quotient is the set $X/\sim$ with plots $U \to X/\sim$ that can locally be written as

$$r \mapsto [\psi(r)],$$

where $\psi : V \to X$ is a plot defined on $V \subseteq U$ and $[\cdot]$ denotes the equivalence class.

This quotient diffeology is the "smallest" diffeology making the quotient map $X \to X/\sim$, $x \mapsto [x]$ smooth.

Remarks on Diffeological Constructions

One can also consider how to construct a diffeological sum $X \sqcup Y$ (Key maps are inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$).

In differential geometry, it is well-known that quotients of manifolds are generally not manifolds themselves. Generally, diffeological spaces are much weaker than manifolds, and can handle quotient spaces.

Construction: Spaces of Maps

The most remarkable construction in diffeology is that the space $\mathcal{C}^\infty(X, Y)$ of smooth maps $X \to Y$ between diffeological spaces is itself a diffeological space!

Definition: Functional Diffeology

A plot for the functional diffeology is any map

$$\Psi : U \to \mathcal{C}^\infty(X, Y), \qquad r \mapsto (\Psi_r : X \to Y)$$

that makes the *evaluation map*

$$\tilde{\Psi} : U \times X \to Y, \qquad (r, x) \mapsto \Psi_r(x)$$

smooth.

Remarks on the Functional Diffeology

The fact that diffeology can handle spaces of maps makes for an extremely *convenient* setting.

Remark:

Even in topology, the space $\mathcal{C}^0(X, Y)$ of *continuous* maps is generally not a topological space in a natural way.

The Diffeology on Path Spaces

In the special case $X = [0, 1]$ and Y any diffeological space, we define

$$\mathcal{P}Y = \mathcal{C}^\infty([0, 1], Y),$$

Definition: Path Spaces

The path space to a diffeological space is the set $\mathcal{P}Y$ equipped with the functional diffeology.

A plot $r \mapsto \alpha_r$ for the path space is viewed as a smooth variation of paths in Y.

Applications of Path Spaces

Shortcuts in Differential Geometry Results

The *homotopy invariance* of de Rham cohomology from differential geometry can be proved using a shortcut through a space which isn't a manifold—the path space.

Geometric Quantization by Paths

Shown in a recent paper (Iglesias-Zemmour 2025), the *prequantum groupoid* from geometric quantization can be constructed as a diffeological quotient of a path space.

Bundles Over Path Spaces

It is possible to study fiber bundles over diffeological spaces (in particular, path spaces). This study leads to the area of *higher gauge theory*, which is an active area of research in mathematics and physics.

References

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