The Universe and Its Internal Dimensions

A SIGMA Presentation

Dion Mann

12 September 2025

Contents

Introduction

Physics as We Understand It

Geometry of the Universe

- Space and time are to be thought of as a single entity.
- Since there are three spatial dimensions and one temporal dimension, our universe is a **four-dimensional manifold.**
- A **manifold** is a topological space with open sets that look like $\mathbb{R}^n$.

Mechanics according to Einstein

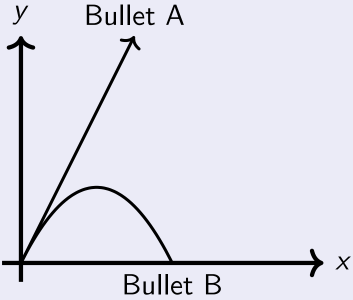
- Paths traced out by massive objects under gravity follow **geodesics.**
- A **geodesic** is a *straightest possible* path.
- Geodesics coincide when they overlap *and* share a tangent vector.

Higher Dimensions?

Why require the temporal dimension?

Example: Two Bullets.

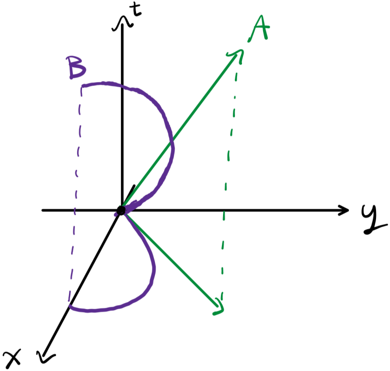
Assume Earth is 1-dimensional. Let (x, y) be co-ordinates, with x along Earth and y directed upwards. Bullet A and Bullet B are shot at the same angle, yet their trajectories are as below (under the influence of gravity). How is this possible?



Higher Dimensions?

Why require the temporal dimension?

- Now let's add the temporal dimension.



- The geodesics didn't coincide…
  - *…because they didn't have the same tangent vector to begin with.*

Higher Dimensions?

Should we require any other dimensions?

- Short answer: yes.
  - In modern physical theory, especially those governing particle interactions, we need more than the standard 4 dimensions.

Example: Kaluza–Klein's electromagnetic theory

The theory of electromagnetism built upon the idea that the model of our universe is $M \times U(1)$ (where M is spacetime and $U(1)$ is the circle $\mathbb{S}^1$).

- Observation: We now have a 5-dimensional manifold, up from 4! (not a factorial)
- Let's refer to this as $U(1)$-theory, for brevity.

- But, if we see 3 dimensions of space and 1 of time, what would the extra dimensions even be?

Internal Spatial Dimensions

Interpretation of extra dimensions

- Physicists interpret the extra dimension (e.g. coming from $\mathbb{S}^1$ in $U(1)$-theory) as an extra *spatial* dimension.
- In other words, we would have 4 spatial and 1 temporal dimension.
- Humans (us) cannot see or interact with the 4th spatial dimension...
  - ...because we interpret the spatial dimension to be physically *tiny.*

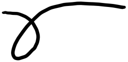
Example: $U(1)$-Theory

- In the case of $U(1)$-theory, the circle $\mathbb{S}^1$ is supposed to be a very small dimension curled up.
- In $M \times U(1)$, every element $x \in M$ determines a space $\{x\} \times U(1)$.
  - We interpret $\{x\} \times U(1)$ as being *internal*.
  - In other words: moving along the circle $\{x\} \times U(1)$ somehow means you haven't left the point x.

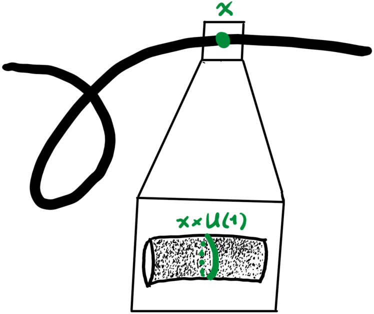
Mathematics of Gardening

Analogy for Internal Dimensions: A Garden Hose

- I learned this analogy from the mathematical physicist Roger Penrose:



- The universe is the apparently 1-dimensional hose. . .
- . . . but upon zooming in, we see a small, curled up 2nd dimension.



Fiber Bundles

The Idea of Bundles

Associating Spaces to Points

- The idea of an entire space *internal* to some point has a beautiful formulation in geometry/topology.
- We need to abstract the ideas in the previous section.

The Topological Picture

1. Let's draw a step-by-step picture on the whiteboard.
2. Start with a *base space* called B and some other space F.
3. At each $x \in B$, consider $\pi^{-1}(x) := \{x\} \times F$. We view $\pi^{-1}(x)$ as a space "sitting above" x.
4. Let $E = \bigcup_{x \in B} \{x\} \times F$ be the collection of all these spaces. In the same sense, it "sits above" B.

Observation: we have a *projection* $\pi : E \to B$ given by $\pi(\{x\} \times F) = \{x\}$.

Fiber Bundles

Formal Definition: Fiber Bundle

A **fiber bundle** is a surjective map $\pi : E \to B$ between manifolds E and B that is *locally trivial.* This means there is a manifold F such that every point $x \in B$ has a neighborhood U and a **diffeomorphism** $T_U : \pi^{-1}U \to U \times F$ such that $\mathrm{pr}_1(T_U(p)) = \pi(p)$.

- A **diffeomorphism** is an isomorphism, but for manifolds.
  - Formally, they are smooth with a smooth inverse.
- In our picture, we had $E = B \times F$ instead of $\pi^{-1}U \cong U \times F$.
  - It is too strong to require the entire space to be a product.
  - We only require it to “look like” a product, zooming in far enough.
- Think of T_U as a tool to write $p \in E$ as a product $(x, \xi) \in B \times F$.
  - Geometers are used to making the identification $p = (x, \xi)$.

Foundations for Kaluza–Klein Electromagnetism

The fiber bundle for $U(1)$-theory

- We can begin to formulate electromagnetic theory already.
- The base is $B = M$, the spacetime manifold.
- The fibers are circles, so $F = U(1)$.
- It is sufficient to take any bundle $\pi : P \to M$ with fibers $U(1)$ of a certain type...
  - ...but more on this later.

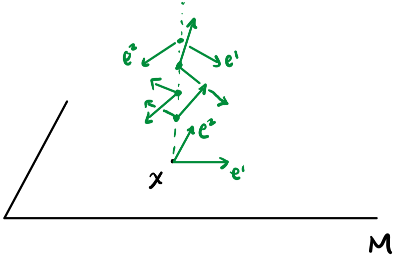
How is $U(1)$-theory extracted from the bundle?

- We have the basic object, but what do we do with it?
- Let's first make a few independent observations:
  - Any *measurement* is made within a **frame of reference.**
  - The act of measurement is part of the system.
  - Therefore, any theory should include this interaction.

A Bundle for Reference Frames

The Frame Bundle

- Every manifold M gives rise to a *bundle of reference frames*.
- Formally, if $x \in M$, we can consider the set of all bases $(e_1, \ldots, e_n)$ for the tangent space $T_xM \cong \mathbb{R}^n$:



- Let $F(M)$ be the collection of all bases for each T_xM.
- The projection sends a basis for T_xM to the footprint $x \in M$, so we have a map $\pi : F(M) \to M$.

Structures on the Frame Bundle

The Fibers of the Frame Bundle

- We have $\pi : F(M) \to M$, but what are the fibers?
- Let $\vec{b}_0 = (v^1, \ldots, v^n)$ be some basis for $T_x M$. If $\vec{b} = (u^1, \ldots, u^n)$ is another basis here, then there is a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\vec{b} = A \cdot \vec{b}_0.$$

- Therefore, $\pi^{-1}(x) \cong \mathrm{GL}_n(\mathbb{R})$.

Fibers and Actions

- The fiber $G := \mathrm{GL}_n(\mathbb{R})$ is not just a space, but also a *group*.
- Furthermore, G acts on $F(M)$ by matrix multiplication.
- The frame bundle $\pi : F(M) \to M$ is an example of a **principal bundle.**

Principal Bundles

Formal Definition: Principal Bundle

A **principal** G**-bundle** is a fiber bundle $\pi : P \to M$ along with a smooth, free action $P \times G \to P$ such that:

1. The fibers are $\pi^{-1}(\pi(p)) = \{p \cdot g : g \in G\} \cong G$.
2. The local trivializations are compatible with the action: more precisely, $T_U : \pi^{-1}U \to U \times G$ has the form

   $$T_U(p) = (\pi(p), s_U(p))$$

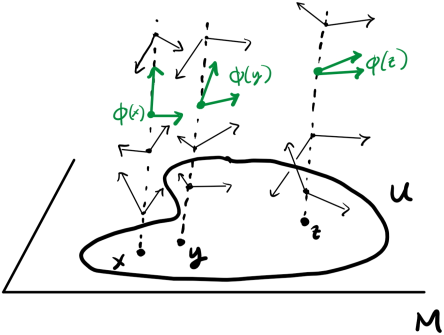
   with $s_U : P \to G$ satisfying $s_U(p \cdot g) = s_U(p) \cdot g$.

Examples.

- The frame bundle $\pi : L(M) \to M$ is a principal G-bundle with $G = \mathrm{GL}_n(\mathbb{R})$.
- The bundles in $U(1)$-theory are principal with $G = U(1)$.

Interpretations of the Frame Bundle

- View $\pi : F(M) \to M$ as a *smooth concatenation of reference frames.*
- Given a local trivialization $T_U : \pi^{-1}U \to U \times G$, we have a natural selection of reference frames for each point $x \in U$:



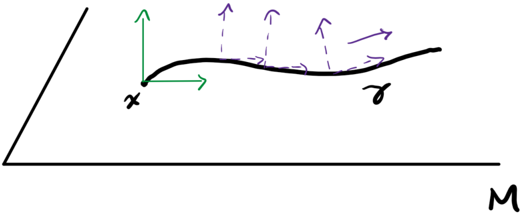
- Define $\phi : U \to F(M)$ by $\phi(x) = T_U^{-1}(x, e)$.
- Since T_U^{-1} is smooth, ϕ is smooth.

Transporting Frames

- There is a similar idea with **transportation** of frames.

Question:

Given a path γ in M starting at x, and a reference frame at x, is it possible to smoothly transport the frame along γ, keeping it as constant as possible?



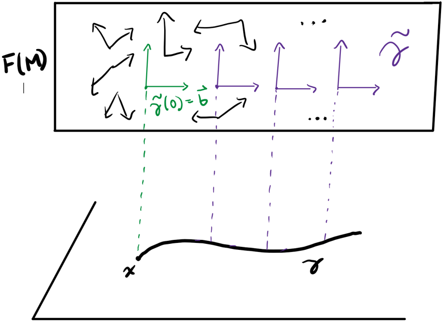
Horizontal Liftings

Alternative View of the Question

- We draw the fibers of a bundle vertically.

Question:

Given a path γ in M starting at x, and a reference frame $\vec{b}$ at x, can we find a lift $\tilde{\gamma}$ into $F(M)$ starting at $\vec{b}$ and is as **horizontal as possible**?

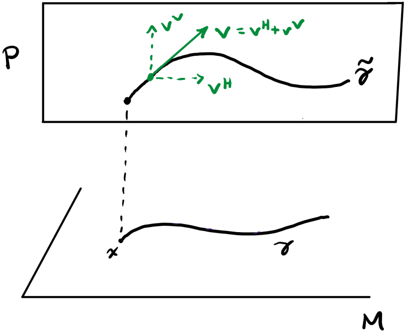


Connections and Physics

Connecting Fibers

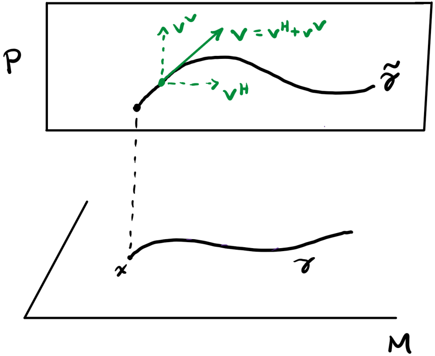
Vertical Projections

- We need a way of telling if a path is *tending vertical* or not.
- In other words: a tangent (direction) vector for a path should have a horizontal and vertical component.



- One way to exhibit this is to have a projection $\omega : T_pP \to T_pP$. Heuristically, $\omega(v^H + v^V) = v^V$.

Connecting Fibers



Telling Horizontal from Vertical

- The idea: any projection allows a decomposition into two subspaces, $\ker \omega$ and $\mathrm{im}\ \omega$.
- We would call $v \in \ker \omega$ **horizontal.**

Connection 1-forms

- Recall that the fibers of a principal G-bundle are the group G.
- So, a vertical projection ω on T_pP should have values in the tangent space of G: let $\mathfrak{g}$ be that space.
- A path $\tilde{\gamma}$ in P would be **horizontal** if ω annihilates all of its direction vectors—there would be no "vertical" components!

Formal Definition: Connection

A **connection** on a principal G-bundle $\pi : P \to M$ is a map $\omega : T_pP \to \mathfrak{g}$ such that:

1. Vertical projection: $\omega(A) = A$ if $A \in \mathfrak{g}$,
2. Compatibility: $\omega(X \cdot g) = g^{-1}\omega(X)g$.

Existence of Connections

Existence of Connections

Theorem

On any principal G-bundle $\pi : P \to M$, a connection exists.

- That means we can always transport data from P along a path in M that is as *horizontal in P* as possible.
  - In other words: as constant as possible.

Implications

- On the frame bundle $\pi : F(M) \to M$, this means we can change reference frames in a consistent way.
  - This allows, for example, the **parallel transport** of vectors in manifolds.
- A natural question: what are the implications of connections on other G-bundles?

Revisiting $U(1)$-Theory

- Let's recall our basic setup for electromagnetic theory.

Kaluza–Klein Electromagnetism

Let M be space-time and $\pi : P \to M$ any principal G-bundle over M with $G = U(1)$. This 5-dimensional PFB is the base object in $U(1)$-theory.

- Since connections exist on any principal G-bundle, let ω be a connection on $\pi : P \to M$.

Theorem

The equations for electromagnetism (**Maxwell's equations**)

$$\begin{aligned} \nabla \cdot E &= \rho & \qquad \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0 & \qquad \nabla \times B &= J + \frac{\partial E}{\partial t} \end{aligned}$$

arise from the connection ω on the $U(1)-$bundle $\pi : P \to M$.

Proving Maxwell's Equations (Sketch)

Proof (Sketch).

- Maxwell's equations take place in spacetime M, so we need to yank the connection down from P to M.
  - This is possible by considering $\omega_U = \sigma_U^* \omega$, where $\sigma_U : M \to P$ could be $\sigma_U(x) = T_U^{-1}(x, e)$.
  - Since ω has values in $\mathfrak{u}(1) = i\mathbb{R}$, we can write $\omega_U = iA_U$, where A_U is $\mathbb{R}$-valued.
- Put $F_U = dA_U$, which is a sort of derivative of A_U.
- Because $U(1)$ is abelian, one can show $dA_U = dA_V$ on overlaps, hence $F_U = F_V$.
- Therefore, each F_U patch together into a global F defined on all of M.
- This F has the form $F = \vec{E} \cdot d\vec{r} \wedge dt + \vec{B} \cdot d\vec{\sigma}$ and satisfies $dF = 0$ and $\delta F = \rho \; dt - \vec{J} \cdot d\vec{r}$.
- The conditions above are precisely Maxwell's equations.

Interpreting the Theory

- There are a couple of interesting things to think about here.

  1. It seems we need 5 dimensions to at least describe electromagnetic interactions in our universe.
     - ⋆ But there are more laws!
     - ⋆ Is it possible to describe all laws with a single bundle?
     - ⋆ If so, how many dimensions do we need?

  2. Maxwell's equations seem to arise on *any* principal bundle over spacetime with fiber $U(1)$.
     - ⋆ We have "abstract electromagnetism" on bundles that don't necessarily describe our universe.
     - ⋆ Are Maxwell's equations just mathematical artifacts necessarily arising from the theory of bundles, or are they physical discoveries due to some fundamental physical principle? Or both?