

# Quantum Mechanics as a Deformation of Classical Mechanics

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# Classical Mechanics

# How to Describe Motion

- Suppose a particle with mass  $m$  is moving on the line  $\mathbb{R}$  (or on the plane  $\mathbb{R}^2$ , etc.)
- Some things we can discuss about the particle:
  - ① Its **position**  $x(t)$  on the line at time  $t$ .
  - ② Its **velocity**  $v(t) = \frac{dx}{dt}(t)$  at time  $t$ .
    - ▶ In physics, we often speak of the particle's **momentum**  $p = m \frac{dx}{dt}$  as much as its velocity.
- The path traced out by  $x(t)$  is called the particle's **trajectory**.

# Forces and Newton's Second Law

- Sometimes all we know is the *forces* acting on the object.
  - ▶ Gravity.
  - ▶ Electrostatic.
  - ▶ Friction.

## Newton's Second Law of Motion

The total force on an object is the product of its mass and acceleration:

$$F = ma(t) \iff F = m \frac{dv}{dt} \iff F = m \frac{d^2x}{dt^2}$$

- Notice that  $\frac{dp}{dt} = \frac{d}{dt} \left( m \frac{dx}{dt} \right) = m \frac{d^2x}{dt^2} = F$ .

## Newton's Second Law of Motion (with momentum)

The total force on an object is the instantaneous change of its momentum:

$$F = \frac{dp}{dt}$$

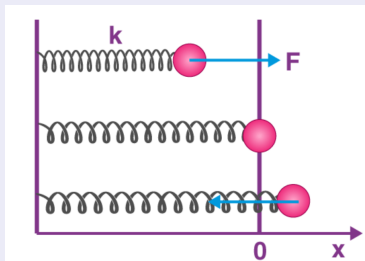
# The Harmonic Oscillator: Introduction

- Our focus will be on a physical system called the **harmonic oscillator**.
  - ▶ This is a **system** where the motion is *oscillatory*.
- There are two types of harmonic oscillators:
  - ▶ The **classical** harmonic oscillator; we will do this now.
  - ▶ The **quantum** analogue... also called the harmonic oscillator.
- The values of measurements are drastically **different** between classical and quantum harmonic oscillators.
  - ▶ The classical harmonic oscillator has **continuous** values of total energy.
  - ▶ The quantum harmonic oscillator has **discrete** values of total energy.

# Classical Harmonic Oscillator

## Example: A Block Attached to a Spring

A block of mass  $m$  attached to a spring is pulled a distance of  $x_0$  away from equilibrium, and is then released *from rest*.



- Using Newton's 2nd law  $F = m \frac{d^2x}{dt^2}$ , we show that the trajectory of the block is given by  $x(t) = x_0 \cos(\omega t)$ .

# Conservation of Energy

- We have  $x(t) = x_0 \cos(\omega t)$  when  $x(0) = x_0$  and  $x'(0) = 0$ .
- There are two types of energy for the harmonic oscillator:
  - ① Kinetic energy: the energy of **motion**,  $K = \frac{1}{2}mv^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m}$
  - ② Potential energy: the energy of **position**,  $U = \frac{1}{2}kx^2$ .
- Consider the total energy of the system:

$$H = K + U = \frac{1}{2}(mv^2 + kx^2) = \frac{p^2}{2m} + U(x),$$

called the **Hamiltonian**.

- ▶ We treat  $x$  and  $p$  as independent variables:  $H = H(x, p)$ .
- ▶ The Hamiltonian is constant throughout time, i.e.  $\frac{dH}{dt} = 0$ .

# Hamiltonian Mechanics

- Generally, the total energy (Hamiltonian)

$$H(x, p) = \frac{p^2}{2m} + U(x)$$

for any system is *conserved*.

- Hamilton used  $H(x, p)$  to shift the focus from forces to **energy**:

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \left( \frac{p^2}{2m} + U(x) \right) = \frac{p}{m} = \frac{dx}{dt}$$

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x} \left( \frac{p^2}{2m} + U(x) \right) = \frac{dU}{dx}$$

# Hamilton's Equations

- What is  $\frac{dU}{dx}$ ?
  - ▶ In harmonic oscillator, we had  $U(x) = \frac{1}{2}kx^2$ , so  $\frac{dU}{dx} = kx$
  - ▶ This almost looks like the spring force  $F = -kx$ .
- In general, potential energy is a function  $U(x)$  such that  $F = -\frac{dU}{dx}$ .
  - ▶ Resuming our calculation,

$$H(x, p) = \frac{p^2}{2m} + U(x) \implies \frac{\partial H}{\partial x} = \frac{dU}{dx} = -F = -\frac{dp}{dt}.$$

## Hamilton's Equations

If  $H = H(x, p)$  is the Hamiltonian of a system, then

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

# Poisson Brackets

- Hamilton's two equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

can be expressed in a unified way.

- The **Poisson bracket**: for  $f(x, p)$  and  $g(x, p)$ , their Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x}.$$

- **Examples:**

- ①  $f(x, p) = x, g(x, p) = p.$
- ②  $f(x, p) = x, g(x, p) = H(x, p).$
- ③  $f(x, p) = p, g(x, p) = H(x, p).$

- Hamilton's equations are both special cases of the single equation

$$\frac{df}{dt} = \{f, H\}.$$

# Quantum Mechanics

# Summary of Classical Setting

- What mathematical objects are used in classical mechanics?
  - ▶ Particles are described by **numbers**  $(x, p)$
  - ▶ Measurements are **real numbers**.
    - ① Position:  $r(x, p) = x$ .
    - ② Velocity:  $v(x, p) = \frac{p}{m}$ .
    - ③ Kinetic energy:  $K(x, p) = \frac{p^2}{2m}$ .
    - ④ Total energy (Hamiltonian):  $H(x, p) = \frac{p^2}{2m} + U(x)$ .
  - ▶ Properties we measure (e.g. position) are called **observables**.
  - ▶ The **time-evolution** of a particle is described by

$$\frac{dx}{dt} = \{x, H\}.$$

# The Quantum Setting

- What mathematical objects are used in quantum mechanics?
  - ▶ Particles are represented by **functions**  $\psi(x)$ .
  - ▶ Observables (properties we measure) are linear operators on functions.
    - ① Position operator:  $\hat{x} : \psi(x) \mapsto x\psi(x)$
    - ② Momentum operator:  $\hat{p} : \psi(x) \mapsto -i\hbar \frac{d\psi}{dx}$   
( $\hbar \approx 10^{-34} \text{ J} \cdot \text{s}$  is called **Planck's constant**)
    - ③ **Hamiltonian**:  $\hat{H}$ . We will discuss this later in an example.
  - ▶ Measurements of a property are **eigenvalues** of its **operator**.
  - ▶ The **time-evolution** of a particle is given by **Schrödinger's equation**:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

where  $\Psi(x, t)$  is the particle's state at time  $t$  and  $\Psi(x, 0) = \psi(x)$ .

# Differences in Algebraic Structure

- Classical observables (functions) have a **commutative** product:

$$fg = gf, \quad \text{so } fg - gf = 0.$$

- Quantum observables (operators) have a **noncommutative** product, given by **composition** of operators.

- ▶ For example, using  $\hat{x}\psi(x) = x\psi(x)$  and  $\hat{p}\psi(x) = -i\hbar \frac{d\psi}{dx}$ :

$$(\hat{x}\hat{p})\psi = \hat{x} \left( -i\hbar \frac{d\psi}{dx} \right) = -i\hbar x \frac{d\psi}{dx}$$

$$(\hat{p}\hat{x})\psi = \hat{p}(x\psi) = -i\hbar \frac{d}{dx}(x\psi) = -i\hbar \left( \psi + x \frac{d\psi}{dx} \right) = -i\hbar x \frac{d\psi}{dx} - i\hbar\psi$$

- ▶ So  $(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = i\hbar\psi$ , so  $\hat{x}\hat{p}$  and  $\hat{p}\hat{x}$  differ by scaling by  $i\hbar$ :

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \neq 0.$$

# Quantum Harmonic Oscillator: Introduction

- The **classical Hamiltonian** for the harmonic oscillator is

$$H(x, p) = \frac{1}{2} (x^2 + p^2)$$

(for simplicity, we won't write  $m$  or  $\omega$ )

- The **quantum Hamiltonian** is obtained by  $x \mapsto \hat{x}$  and  $p \mapsto \hat{p}$  in  $H$ , where  $\hat{x}\psi(x) = x\psi(x)$  and  $\hat{p}\psi(x) = -i\hbar \frac{d\psi}{dx}$ . Set

$$\hat{H} = \frac{1}{2} (\hat{x}^2 + \hat{p}^2)$$

where

$$\hat{x}^2(\psi(x)) = \hat{x}(\hat{x}(\psi(x))) = x^2\psi(x)$$

$$\hat{p}^2(\psi(x)) = -i\hbar \frac{d}{dx} \left( -i\hbar \frac{d\psi}{dx} \right) = (-i\hbar)^2 \frac{d^2\psi}{dx^2}$$

# Energy Values of the Quantum Harmonic Oscillator

- Recall the total energy of a **classical** harmonic oscillator is

$$H(x, p) = \frac{1}{2} (x^2 + p^2).$$

Its values are all numbers  $\geq 0$ .

- What about the **quantum** harmonic oscillator?

$$\hat{H} = \frac{1}{2} (\hat{x}^2 + \hat{p}^2)$$

- Measurements are *eigenvalues* of operators: the energy values of the quantum harmonic oscillator are the eigenvalues of  $\hat{H}$ .

## Energy Levels of Quantum Harmonic Oscillator

The possible energies (eigenvalues) of  $\hat{H}$  turn out to be

$$\left(n + \frac{1}{2}\right) \hbar \geq \frac{1}{2} \hbar \quad n = 0, 1, 2, \dots$$

# Eigenfunctions of the Quantum Harmonic Oscillator

## Eigenvalues and Eigenfunctions for $\hat{H}$

The function  $\psi_n(x) = e^{-x^2/2\hbar} H_n(x/\sqrt{\hbar})$  satisfies  $\hat{H}\psi_n = \left(n + \frac{1}{2}\right) \hbar \cdot \psi_n$ , where  $H_n$  are the *Hermite polynomials*.

Proof (Only for  $n = 0$ ).

- Consider  $\psi_0(x) = e^{-x^2/2\hbar}$ . We have  $\hat{H}\psi_0 = \frac{1}{2} (x^2\psi_0 - \hbar^2\psi_0'')$ .
- Check that  $\hbar^2\psi_0'' = x^2\psi_0 - \hbar\psi_0$ .
- Then  $\hat{H}\psi_0 = \frac{1}{2}\hbar\psi_0$ .

□

# Disjoint Theories of Physics

- Consider how **different** classical and quantum mechanics are.
  - ▶ Continuous vs. discrete energies:  $E \geq 0$  vs.  $E_n = \left(n + \frac{1}{2}\right) \hbar$ .
  - ▶ The lowest classical energy level is  $H(0,0) = 0$ , the lowest quantum energy level is  $E_0 = \frac{1}{2}\hbar > 0$ .
    - ★ But, note  $\frac{1}{2}\hbar \rightarrow 0$  as  $\hbar \rightarrow 0$ .
- This is related to fundamentally different mathematical objects.
  - ▶ Classical mechanics uses functions, quantum mechanics uses operators.
  - ▶ Products of functions commute, products of operators don't.
- Can a **single theory** describe both classical and quantum mechanics?
  - ▶ One attempt is known as **deformation quantization**, which considers **only functions** (no operators!) using a noncommutative product on functions involving  $\hbar$  that becomes ordinary product when  $\hbar \rightarrow 0$ .

# Deformation Quantization

# The Deformation Philosophy

- New physics can be viewed as a **deformation** of old physics.

## Example: Shape of Earth

- We (humans approximately 5-6ft tall) experience similar physics whether...
  - 1 The earth is **flat**.
  - 2 The earth is a big **sphere**.
- The sphere theory **deforms** to the flat theory in the limit where we zoom in very closely (spheres are locally Euclidean).
- We'll view **quantum theory as a deformation of classical theory**. We want formulas with  $\hbar$  to become classical formulas when  $\hbar \rightarrow 0$ .

# Deforming Products

## Main Idea

If we only use functions, let's mimic the noncommutative nature of quantum operators by a **noncommutative product** on functions.

- We want a product  $\star$  on functions  $f(x, p)$  using  $\hbar$  such that
  - ①  $\star$  is noncommutative
    - In particular:  $x \star p - p \star x = i\hbar$ , just like for  $\hat{x}$  and  $\hat{p}$  before.
  - ② If  $\hbar \rightarrow 0$ , then  $f \star g \rightarrow f \cdot g$ , where  $\cdot$  is the usual product.
- Viewing  $\hbar$  as a **small parameter**, we deform  $\cdot$  by having  $f \star g$  be a power series in  $\hbar$  with coefficients depending on  $f$  and  $g$  :

$$f \star g = fg + \mu_1(f, g)\hbar + \mu_2(f, g)\hbar^2 + \dots$$

As  $\hbar \rightarrow 0$ ,  $f \star g \rightarrow fg$ . Call this  $\star$  a **star-product**.

# Properties of Star-Products

- We want

$$f \star g = fg + \mu_1(f, g)\hbar + \mu_2(f, g)\hbar^2 + \dots,$$

where  $f = f(x, p)$  and  $g = g(x, p)$ . What properties should the coefficients  $\mu_n(f, g)$  have?

- We want  $x \star p - p \star x = i\hbar$ .
  - ▶ Then  $\mu_1(x, p) - \mu_1(p, x) = i$ .
  - ▶ Compare to  $\{x, p\} = 1$ , where  $\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x}$ .

## Star-Product Axioms

For all functions  $f(x, p)$ ,  $g(x, p)$ , and  $h(x, p)$ :

- **Classical limit:**  $\mu_1(f, g) - \mu_1(g, f) = i\{f, g\}$ .
- **Associativity:**  $f \star (g \star h) = (f \star g) \star h$ .

## Example of a Star-Product: The Basic Star-Product

- Recall for classical harmonic oscillator  $H(x, p) = \frac{1}{2}(x^2 + p^2)$ .
- In  $\mathbb{C}$ , we factor  $H$  by letting  $a = \frac{1}{\sqrt{2}}(x + ip)$  and  $b = \frac{1}{\sqrt{2}}(x - ip)$ :
  - ▶  $H = \frac{1}{2}(x^2 + p^2) = \frac{1}{\sqrt{2}}(x + ip) \cdot \frac{1}{\sqrt{2}}(x - ip) = ab$ .
  - ▶ So we can view  $H$  as a function of  $a$  and  $b$ :  $H(a, b) = ab$ .
- In general, think of functions  $f(x, p)$  as functions  $f(a, b)$ .
  - ▶ Like  $(x, y) \rightarrow (r, \theta)$  in polar coordinates.

### The basic star-product

Define the **basic star-product** by

$$f \star g = fg + \frac{\partial f}{\partial a} \frac{\partial g}{\partial b} \hbar + \frac{\partial^2 f}{\partial a^2} \frac{\partial^2 g}{\partial b^2} \frac{\hbar^2}{2} + \dots + \frac{\partial^n f}{\partial a^n} \frac{\partial^n g}{\partial b^n} \frac{\hbar^n}{n!} + \dots$$

## Examples with basic star-product

- Let's do some examples with the basic star-product:

$$f \star g = fg + \frac{\partial f}{\partial a} \frac{\partial g}{\partial b} \hbar + \frac{\partial^2 f}{\partial a^2} \frac{\partial^2 g}{\partial b^2} \frac{\hbar^2}{2} + \dots + \frac{\partial^n f}{\partial a^n} \frac{\partial^n g}{\partial b^n} \frac{\hbar^n}{n!} + \dots$$

where  $a = \frac{1}{\sqrt{2}}(x + ip)$  and  $b = \frac{1}{\sqrt{2}}(x - ip)$ .

### Example 1

Note that  $x = \frac{\sqrt{2}}{2}(a + b)$  and  $p = \frac{\sqrt{2}}{2i}(a - b)$ . Calculate  $x \star p$  and  $p \star x$ .

### Example 2

Let  $H(a, b) = ab$  and  $g = g(a, b)$  be any function. Show that

$$H \star g = Hg + \hbar b \frac{\partial g}{\partial b}.$$

# The Star-Schrödinger Equation

- Recall **quantum mechanics** is controlled by Schrödinger's equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

where an operator  $\hat{H}$  acts on the function  $\Psi(x, t)$ .

- In deformation quantization, we have functions  $\psi = \psi(a, b)$  and the  $\star$ -product, but no operators.

## Star-Schrödinger Equation

Let  $H = H(a, b)$  be the *classical* Hamiltonian function. Then the quantum time-evolution of  $\psi = \psi(a, b)$  is given by the **star-Schrödinger equation**:

$$i\hbar \frac{\partial \psi}{\partial t} = H \star \psi$$

# Deformation Quantization of the Harmonic Oscillator

- Recall the **classical harmonic oscillator** is described by the Hamiltonian

$$H(x, p) = \frac{1}{2} (x^2 + p^2) \quad \text{or} \quad H(a, b) = ab,$$

where  $a = \frac{1}{\sqrt{2}}(x + ip)$  and  $b = \frac{1}{\sqrt{2}}(x - ip)$ .

- This represents the **total energy** of the harmonic oscillator; it takes continuous values in  $[0, \infty)$ .
- Using a suitable star-product, the Hamiltonian function  $H(a, b)$  can give us the discrete **energy levels** of the quantum harmonic oscillator by a type of eigenvalue.
  - Call a function  $g(a, b)$  a **star-eigenfunction** of  $H(a, b)$  if there is a number  $\lambda$ , called the **star-eigenvalue**, such that  $H \star g = \lambda g$ .
  - We want the star-eigenvalues of the Hamiltonian for classical harmonic oscillator  $H$  to be the energy levels of quantum harmonic oscillator.

# Energy Levels of the Harmonic Oscillator

## Eigenvalues of $H(a, b) = ab$

For each  $n \in \mathbb{Z}_{\geq 0}$ , let  $g_n(a, b) = (ab)^n e^{-ab/\hbar}$ . Then

$$H \star g_n = (n\hbar)g_n,$$

where  $f \star g = fg + \frac{\partial f}{\partial a} \frac{\partial g}{\partial b} \hbar + \frac{\partial^2 f}{\partial a^2} \frac{\partial^2 g}{\partial b^2} \frac{\hbar^2}{2} + \dots$  is the basic star-product.

Proof (Outline).

- By a previous example,  $H \star g = Hg + \hbar b \frac{\partial g}{\partial b}$  for any  $g = g(a, b)$ .
- Take  $g = g_n$ , and use the product rule in the **second term** to get

$$H \star g_n = (ab)^{n+1} e^{-ab/\hbar} + n\hbar(ab)^n e^{-ab/\hbar} - (ab)^{n+1} e^{-ab/\hbar}$$

- Simplify:  $H \star g_n = n\hbar g_n$ .



# Discussion: Energy Levels of the Harmonic Oscillator

- Let  $\psi_n(a, b, t) = g_n(a, b)e^{-int}$ .
  - ▶ Then  $i\hbar \frac{\partial \psi_n}{\partial t} = i\hbar (-in \cdot g_n e^{-int}) = n\hbar \psi_n$ .
  - ▶ On the other hand,  $H \star \psi_n = H \star (g_n e^{-int}) = n\hbar g_n e^{-int} = n\hbar \psi_n$ .
- So  $\psi_n$  solves the star-Schrödinger equation  $i\hbar \frac{\partial \psi_n}{\partial t} = H \star \psi_n$ .
- But there is a problem:
  - ▶ We found the energy levels to be  $n\hbar$ , for  $n = 0, 1, 2, \dots$
  - ▶ Standard quantum mechanics say  $\left(n + \frac{1}{2}\right) \hbar$ , for  $n = 0, 1, 2, \dots$
- A different star-product, called the **Moyal star-product**, corrects this:

$$f \star g = fg + \{f, g\} \frac{(\hbar/2)}{1!} + \left( \frac{\partial^2 f}{\partial a^2} \frac{\partial^2 g}{\partial b^2} - 2 \frac{\partial^2 f}{\partial a \partial b} \frac{\partial^2 g}{\partial a \partial b} + \frac{\partial^2 f}{\partial b^2} \frac{\partial^2 g}{\partial a^2} \right) \frac{(\hbar/2)^2}{2!} + \dots$$

# Physics according to Star-Products

- Using functions like the classical Hamiltonian  $H(a, b)$  and a suitable star-product, we obtain the energy levels of the quantum harmonic oscillator as star-eigenvalues of  $H$ .
  - ▶ Can match the ordinary eigenvalues of  $\hat{H}$  in standard quantum mechanics.
- The **basic star-product** gave us  $E_n = n\hbar$ , which did not include the shift of  $\frac{1}{2}$ , but the **Moyal star-product** corrects this shift.
- This leads us to an interesting question:

## Uniqueness of Quantum Mechanics

Different star-products determine different physics. What is the mathematical and physical significance of each one?