Quantum Mechanics as a Deformation of Classical Mechanics

Dion Mann [1]

[1]University of Colorado, Boulder

Introduction

With each new discovery, physics often finds itself being forced to radically modify its theories to remain consistent with the physical world. In all these cases, the new physical theory necessarily yields the original models predictions in some limiting behavior (a prototypical example is the ancient flat Earth deformed through modern manifold theory). Thus we can view the emergence of new physics as a deformation of old physics.

In this sense, we can interpret quantum mechanics as a deformation of classical mechanics. The "deformed quantum theory" is measured by $\hbar$, whose limiting behavior $\hbar \to 0$ should represent a contraction back to classical theory. This is the theory of deformation quantization.

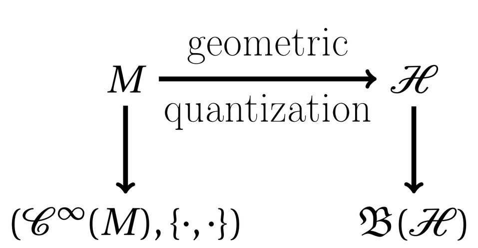
What is Quantization?

Quantization is the process of translating from a classical system to a quantum mechanical system. If this sounds ambiguous, it's because it is. Indeed, there are many approaches to quantize a given classical system.

System	Phase Space	Observables
Classical	Poisson Manifold M (e.g., $T^*\mathbb{R}^n$)	Poisson Algebra $(\mathscr{C}^\infty(M), \{\cdot,\cdot\})$
Quantum	Hilbert Space $\mathscr{H}$ (e.g., $L^2(\mathbb{R}^n)$)	C^*-Algebra $\mathfrak{B}(\mathscr{H})$

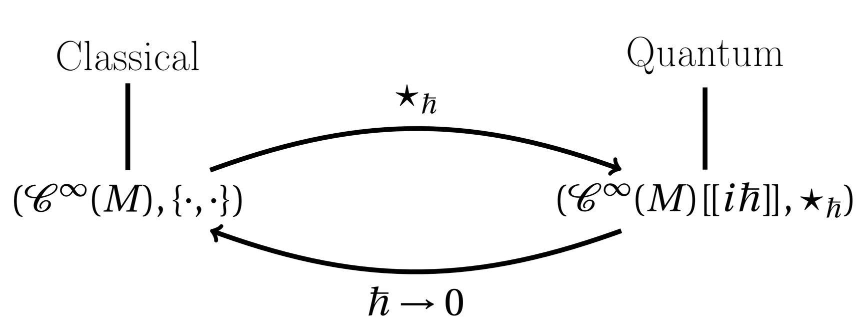
Table 1. A physical system consists of two pieces of information: a phase space and a collection of observables. Observe that the mathematics describing classical mechanics is significantly different than that describing quantum mechanics.

One approach is to directly construct a Hilbert space $\mathscr{H}$ out of a classical system, and then mapping classical observables H to operators $\hat{H}$ on $\mathscr{H}$.



However, this (geometric) quantization approach comes with the caveat that a Hilbert space must be constructed.

In contrast, deformation quantization attempts to deform the algebraic structure of classical observables into that of quantum observables. This way, quantum mechanics can be done without any reference to Hilbert spaces, purely working with quantized observables.



The idea is to deform the standard commutative product $\cdot$ on $\mathscr{C}^\infty(M)$ into a noncommutative product $\star_\hbar$ in such a way that some desired "quantum" properties are satisfied. For example, if $\Omega : \mathscr{C}^\infty(M) \to \mathfrak{B}(\mathscr{H})$ is a quantization in the sense of Dirac (e.g., the Weyl application), we would like $\Omega(f \star_\hbar g) = \Omega(f)\Omega(g)$. Given such a star-product, we can "forget" about the operator algebra and study quantum mechanics through $\star_\hbar$ on a classical system.

Definition: Deformation Quantization of Manifolds

Let (M,π) be a Poisson manifold with $\mathscr{A} = (\mathscr{C}^\infty(M), \{\cdot,\cdot\})$ its Poisson algebra of smooth functions. A deformation quantization of M is a sequence $(C_r)_{r\geq 0}$ of $\mathbb{C}$-bilinear maps $C_r : \mathscr{A}^2 \to \mathscr{A}$ with the following properties:

1. Locality: each C_r is a bidifferential operator.
2. Classical Limits: $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = i\{f,g\} = i\pi(df,dg)$
3. Unity: $C_r(1,f) = 0 = C_r(f,1)$ for all $r \geq 1$.
4. Formal Associativity: $\sum_{s=0}^{r}\left[C_s(C_{r-s}(f,g),h) - C_r(f, C_{r-s}(g,h))\right] = 0$ for all $r \geq 0$.

The deformed product $\star_\hbar = \sum_{r\geq 0} \hbar C_r$, $(f,g) \mapsto f \star_\hbar g = \sum_{r\geq 0} \hbar C_r(f,g)$ is called a star-product on M.

The star-product introduced in the previous section with the property that $\Omega(f \star_\hbar g) = \Omega(f)\Omega(g)$ is called the Moyal–Weyl product. In 1949, Moyal calculated an explicit formula

$$f \star_\hbar g = \exp\left(\frac{i\hbar}{2}\pi\right)(df,dg) = fg + \frac{i\hbar}{2}\{f,g\} + \cdots \qquad (1)$$

A natural question to ask is what manifolds admit a deformation quantization. Fortunately from the perspective of physics, it was proven in 1983 by M. DeWilde and P. Lecomte that a star-product exists on every symplectic manifold (most classical phase spaces). The pinnacle of the hunt for existence was in 1997 when M. Kontsevich proved his "formality conjecture."

The Formality Theorem

Let M be a Poisson manifold. Then there exists an L_∞-algebra quasi-isomorphism between the DGLA of multidifferential operators on M and the DGLA of multivector fields on M.

The formality conjecture implies the existence of a star-product on any Poisson manifold, a much larger class of manifolds than the symplectic case. With the existence of star-products better understood, it is instructive to outline the deformation procedure through a well-known system:

Example: 1-Dimensional Harmonic Oscillator.

The 1-dimensional (classical) harmonic oscillator is a physical system with phase space $M = \mathbb{R}^2$. Coordinatized by (p,q), M is a Poisson manifold whose Poisson bivector is given by $\pi = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$. The Hamiltonian describing the system is the smooth function

$$(p,q) \mapsto H(p,q) = \frac{1}{2}(p^2 + q^2).$$

Choosing the Moyal–Weyl star-product $\star$ (1), we form the star-exponential of H:

$$\exp\left(\frac{Ht}{i\hbar}\right) = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n (\underbrace{H \star \cdots \star H}_{n-\text{times.}}) \qquad (2)$$

and find $\pi_\lambda \in \mathscr{C}^\infty(M)$ such that

$$\exp(Ht/i\hbar) = \sum_{\lambda \in I} \pi_\lambda e^{\lambda t / i\hbar}.$$

This is reminiscent of the spectral theorem in Hilbert space theory. Indeed, I is interpreted as the spectrum of H while the π_λ are the projectors of H. In our case, one finds that

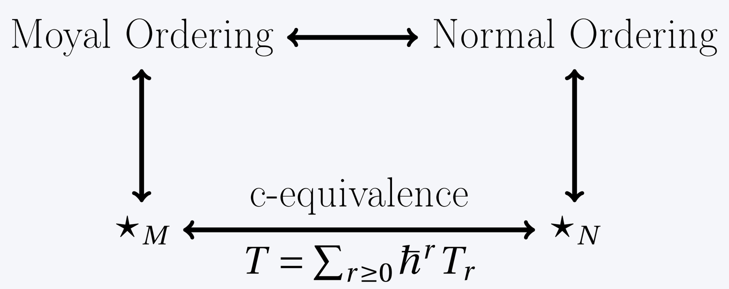
$$\pi_{\lambda = E_n}(p,q) = (-2)^n \exp\left(-\frac{2}{\hbar}H(p,q)\right) L_n\left(\frac{4}{\hbar}H(p,q)\right)$$

with L_n being the Laguerre polynomial of degree n. This eventually leads to the well-known energy levels $E_n = (n + \frac{1}{2})\hbar$, as expected.

Current Research: A Star-Product Approach to Quantum Field Theory

More recently, the question of whether deformation quantization can be applied to field theory was investigated. In deformation quantization à la J. Dito, a star-quantization of the free scalar Hamiltonian can be achieved by considering its star-exponential (2).

The idea is to consider c-equivalent star-products (meaning that there exists a formal series $T = \sum_{r\geq 0} \hbar T_r$ of endomorphisms such that $T(f \star g) = T(f) \tilde{\star} T(g)$). It turns out that c-equivalent star-products correspond to operator ordering in quantum field theory.



A star-product formally equivalent to Moyal's (1) was presented by Dito via the c-equivalence operator

$$T = \exp\left(-\frac{\hbar}{2}\int \frac{\delta^2}{\delta a(k)\delta a^*(k)}\right)$$

and defining $F \star_N G = T^{-1}(T(F) \star_M T(G))$, which corresponds to normal ordering. One then checks that the star-exponential (using the "normal" star-product now) of the free scalar Hamiltonian exists and proceeds to quantization as before.

Some Questions about the Deformation Quantization of Fields

The deformation quantization of fields is ongoing research. As with any new idea, there is the question of finding a "complete" and rigorous mathematical explanation. Considering this leads to a number of questions, a few of which are listd here:

1. There are many star-products (c-equivalent or not) that give the same spectral theory of a given Hamiltonian. What are the implications of this, mathematically or physically?
2. Is it possible to classify star-products that lead to the same star-exponential of a given Hamiltonian? How about for a given family of observables?
3. Given all star-products that lead to the same spectral theory for a free Hamiltonian, does there exist one whose corresponding star-exponential exists for an interacting field?

Acknowledgements

I'd like to thank my undergraduate thesis advisor, Dr. Markus Pflaum, for introducing me to the theory of deformation quantization as well as for reviewing this poster.

I'd also like to thank Dr. Padi Fuster for the Math For All conference, as well as for the many discussions we had about this poster and how to improve it.

References

[1] Joseph Dito. Star-product approach to quantum field theory: The free scalar field. Letters in Mathematical Physics, 20:125 – 134, 1990.

[2] Rui Loja Fernandes. Deformation quantization and poisson geometry. Resenhas, 4(3):327–361, 2000.

[3] M. Flato and D. Sternheimer. Deformations of poisson brackets, separate and joint analyticity in group representations, nonlinear group representations and physical applications. In J. Wolf, M. Cahen, and M. De Wilde, editors, Harmonic Analysis and Representations of Semisimple Lie Groups, chapter 4, pages 419–428. Oxford University Press, 1977.

dion.mann@colorado.edu