Deformation Quantization and Applications to Quantum Field Theory

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Abstract. This is an honors thesis under the Department of Mathematics. The purpose of this paper is to outline the primary concepts in deformation quantization according to [2], including some advancements in its application to quantum field theory (QFT). Therefore we emphasize viewpoints in the "nice" (smooth finite-dimensional manifolds) case of deformation quantization that somewhat survive the field-theoretic context, such as (Hochschild) cohomology and c-equivalence operators. The deformation quantization of fields remains an active area of research, which is acknowledged by this paper and therefore only main ideas are outlined here. Many (outlines of) proofs appearing in the literature also appear in this paper, but full details are provided.

1. Introduction

Quantization, roughly, is a recipe for translating the classical description of a physical system to some corresponding quantum mechanical version. This is hardly a definition, which is both a blessing and a curse. On one hand, this allows for a variety of rich theories of quantum mechanics, each having their own unique approaches to formulating such a recipe. Comparing them can help better our understanding of the physics of our universe and its underlying mathematical structure. On the other hand, the ambiguity of the nature of quantization makes this an extremely difficult area of study. One balances the amount of rigor of a quantum theory in a case-by-case basis. Too little rigor may allow for flexibility in physical predictions, but will surely be mathematical nightmare. Too much rigor can result in a beautiful mathematical theory, yet may find itself too unwieldy to be practically applied.

The purpose of this paper is to study one such approach to quantization: the *deformation* approach. The mathematical toolbox for deformation quantization was slowly assembled overtime until its introduction in the seminal papers of [2, 3] in the 70s. Here it was suggested that quantization be viewed as a deformation—in the sense of Gerstenhaber's deformation theory of associative algebras—of classical observables. As we will see, the mathematical structures of classical and quantum theory are somewhat disjoint. This is what makes the deformation approach especially interesting; given drastically different mathematical structures, there exists a notion of "perturbing" one theory into the other in such a way that the original theory may be recovered in some limit.

We organize the paper as follows: in section 1, we discuss the mathematical theories of classical and quantum mechanics before addressing some quantization schemes. Section 2 introduces the language of deformation according to Gerstenhaber, which we will use to formulate what we mean by *deformation quantization* and give explicit examples. Finally section 3 discusses some directions in quantum field theory, originating from J. Dito's star-product approach [10].

We assume knowledge of some differential geometry and elementary Hilbert space theory.

2. Mathematical Foundations of Physics

In order to make the above ideas precise, we need to work directly with the mathematical backbones of physics. In both classical and quantum theory, the key ingredients to physical systems are the *phase space* and the *observables*. Elements of the phase spaces are called *pure states*.

A **classical phase space** is a real $\mathscr{C}^\infty$-manifold W (connected and paracompact) on which there exists a closed nondegenerate 2-form $\omega \in \Omega^2(W)$ called a symplectic form. The pair (W, ω) is said to be a symplectic manifold. The **classical observables** on W are real-valued $\mathscr{C}^\infty$-functions on W, which form the commutative $\mathbb{R}$-algebra $\mathscr{C}^\infty(W)$. This means that the arena of classical mechanics is symplectic, or more generally as we will soon see, Poisson geometry. On the other hand, a **quantum phase space** is a projective (complex) Hilbert space $\mathbb{P}\mathscr{H}$, in other words the set of 1-dimensional linear subspaces of $\mathscr{H}$. The **quantum observables** consist of self-adjoint operators on $\mathscr{H}$. Thus quantum mechanics is governed by Hilbert space theory.

The purpose of this section is not to give a detailed study of Poisson geometry and Hilbert space theory. Instead, it summarizes some ideas and examples relevant to deformation quantization. The reader may skip this section entirely to section 2, if the appropositionriate background is already there.

2.1. Classical Mechanics and Poisson Geometry. First we recall the notion of a Poisson algebra. Let $\mathscr{A}$ be a unital associative algebra over a commutative ring R. Then $\mathscr{A}$ is said to be a **Poisson algebra** if it admits a Lie bracket $\{\cdot,\cdot\}$ satisfying the *Leibniz identity*, that is $\{x,\cdot\} : \mathscr{A} \to \mathscr{A}$ is a derivation (with respect to the product on $\mathscr{A}$) for all $x \in \mathscr{A}$. Such a bracket is called a **Poisson bracket** on $\mathscr{A}$.

Now let (W, ω) be a symplectic manifold of dimension $2n$. Since ω is nondegenerate, it induces a bundle isomorphism $\hat{\omega} : TW \to T^*W$ given by interior multiplication $X \mapsto i_X\omega$. This allows us to define for each $f \in \mathscr{C}^\infty(W)$ a smooth vector field X_f on W by $X_f = \hat{\omega}^{-1}(df)$, or equivalently by the implicit formula $i_{X_f}\omega = df$, called the **Hamiltonian vector field** associated to f.

PROPOSITION 2.1. *The symplectic form ω induces a Poisson algebra structure on $\mathscr{C}^\infty(W)$.*

PROOF. Define a bracket on $\mathscr{C}^\infty(W)$ by

$$\{f, g\} := \omega(X_f, X_g) = X_g(f) = df(X_g) \tag{2.1}$$

Obviously $\{\cdot,\cdot\}$ is $\mathbb{R}$-bilinear and skew-symmetric, so we first verify the Jacobi identity. Now Cartan's magic formula implies

$$\mathscr{L}_{X_g}(\omega) = d(i_{X_g}\omega) + i_{X_g}(d\omega) = d(dg) + i_{X_g}(d\omega) = 0,$$

since dg and ω are closed. Therefore, for any $X \in \mathfrak{X}(W)$ the following holds:

$$\begin{aligned} 0 = (\mathscr{L}_{X_g}\omega)(X_f, X) &= X_g(\omega(X_f, X)) - \omega(X_f, [X_g, X]) - \omega([X_g, X_f], X) \\ &= X_g(df(X)) - df([X_g, X]) - \omega([X_g, X_f], X) \\ &= X_g X(f) - X_g X(f) - X X_g(f) - \omega([X_g, X_f], X) \\ &= -X(\{f, g\}) - \omega([X_g, X_f], X) \end{aligned}$$

In particular $\omega(X_{\{f,g\}}, X) - \omega([X_f, X_g], X) = 0$, hence by nondegeneracy we have the propositionerty $X_{\{f,g\}} = -[X_f, X_g]$. This allows us to calculate

$$\begin{aligned} \{f, \{g, h\}\} &= X_{\{g,h\}}(f) = -[X_g, X_h](f) = -X_g X_h(f) + X_h X_g(f) \\ &= -X_g(\{f, h\}) + X_h(\{f, g\}) = -\{g, \{h, f\}\} - \{h, \{f, g\}\}, \end{aligned}$$

verifying that $\{\cdot,\cdot\}$ is a Lie bracket on $\mathscr{C}^\infty(W)$. Finally for any $f \in \mathscr{C}^\infty(W)$ we have

$$\{f, gh\} = -\{gh, f\} = -d(gh)(f) = -(hdg + gdh)(f) = \{f, g\}h + g\{f, h\},$$

which verifies the Leibniz property. This shows that (2.1) defines a Poisson bracket on $\mathscr{C}^\infty(W)$.

□

It is sometimes helpful in differential geometry to work in coordinates. We recall a theorem by Darboux that guarantees the existence of charts $(U, (q^i, p_i))_{1\le i\le n}$ in which ω has the form $\omega = \sum_{i=1}^n dq^i \wedge dp_i$. In those Darboux charts, it is a straightforward computation to show that

$$X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

which means that the Poisson bracket (2.1) has the local form

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \tag{2.2}$$

If $W = \mathbb{R}^{2n}$ with canonical global coordinates (q^i, p_i), the Poisson bracket defined by (2.2) is sometimes called the **canonical Poisson bracket** on Euclidean space.

DEFINITION 2.2. Let M be any $\mathscr{C}^\infty$-manifold. If $\mathscr{C}^\infty(M)$ admits a Poisson bracket, then M is said to be a **Poisson manifold**.

This discussion has shown that every classical phase space is in fact a Poisson manifold. Thus the study of classical mechanics is most generally rooted in *Poisson geometry.* There are, of course, Poisson manifolds which do not admit a symplectic structure. One asks how to define a Poisson bracket on the algebra of $\mathscr{C}^\infty$-functions if this is the case. A given Poisson manifold gives rise to a bivector field $\pi \in \Gamma(\bigwedge^2 TM)$ defined by $\pi(df, dg) = \{f, g\}$, so one approach to this question is to consider the other direction: given a bivector field π on M, when does it induce a Poisson bracket on $\mathscr{C}^\infty(M)$?

PROPOSITION 2.3. *The bracket $\{\cdot,\cdot\}$ on $\mathscr{C}^\infty(M)$ defined by*

$$\{f, g\} = \pi(df, dg)$$

is Poisson if and only if $[\pi, \pi]_S = 0$, where $[\cdot,\cdot]_S$ is the Schouten–Nijenhuis bracket [5].

Discussion of the Schouten–Nijenhuis bracket will lead outside the scope of this paper, so we refer the reader to [5] for more details. For our purposes we mention that the Jacobi identity holds for $\{f, g\} = \pi(df, dg)$ if and only if $[\pi, \pi]_S = 0$, while the other three properties always hold. Thus we may equivalently specify a Poisson manifold to be the pair (M, π), where π is such a bivector field on M called a **Poisson tensor** on M.

DEFINITION 2.4. Let (W, ω) be a symplectic manifold together with a distinguished smooth function $H \in \mathscr{C}^\infty(W)$. Then the triple (W, ω, H) is called a **Hamiltonian system** and H is called its **Hamiltonian.**

In physics, the Hamiltonian describes the total energy of the Hamiltonian system (the sum of kinetic and potential energies). The dynamics are specified by the integral curves of X_H, which are curves $(q^i(t), p_i(t))$ such that

$$\begin{cases} \dfrac{dq^i}{dt} = X_H(q^i) = \{q^i, H\} = \dfrac{\partial H}{\partial p_i}. \\ \dfrac{dp_i}{dt} = X_H(p_i) = \{p_i, H\} = -\dfrac{\partial H}{\partial q^i}. \end{cases} \tag{2.3}$$

The system (2.3) is called **Hamilton's equations**. An observable $f \in \mathscr{C}^\infty(W)$ of a Hamiltonian system (W, ω, H) is said to be **conserved** if $\{f, H\} = 0$, the Hamiltonian itself being an obvious example (this is how e.g., the *law of conservation of energy* is derived, which is a crucial principle in solving physical problems).

We end this section with an important example. Suppose a single classical particle freely moves in an n-dimensional $\mathscr{C}^\infty$-manifold M. The coordinates (q^i) on M describe the position of the particle, hence M is sometimes called a **configuration space**. The cotangent bundle $\tau : T^*M \to M$ carries a natural symplectic structure as follows: a point in T^*M has the form (q, α) for some $q \in M$ and $\alpha \in T^*_q M$. We define the **tautological 1-form** $\theta \in \Omega(T^*M)$ by $\theta_{(q,\alpha)} = d\tau^*_{(q,\alpha)}\alpha$. In a canonical chart $(U, (q^i, p_i))$ of (q, α) in T^*M, θ takes the form

$$\theta_{(q,\alpha)} = \sum_{i=1}^n p_i dq^i.$$

Exterior differentiation yields a closed 2-form $\omega = -d\theta \in \Omega^2(T^*M)$. Under the same coordinates, we have $\omega = \sum_{i=1}^n dq^i \wedge dp_i$, hence ω defines a symplectic form on T^*M. In other words, (T^*M, ω) becomes the phase space of the system. The fibres $T^*_q M$ physically represent the particle's momenta conjugate to q. Perhaps unsurprisingly, cotangent bundles are therefore the most common phase spaces in classical mechanics.

For more on classical mechanics, see [28] as well as [1].

2.2. Quantum Mechanics and Spectral Theory. We recall some notions from Hilbert space theory for Schrödinger's quantum mechanics, cf. sections 7–9 in V. Moretti's book "Spectral Theory and Quantum Mechanics" [21]. Let $\mathscr{H}$ be a complex Hilbert space and $(\Omega, \mathscr{A})$ a measurable space. Let $\mathrm{proj}(\mathscr{H})$ be the set of projections on $\mathscr{H}$, that is the set of self-adjoint operators A on $\mathscr{H}$ such that $A^2 = A$. Recall that a **spectral measure** E (or a projection-valued measure) on Ω is a map $E : \mathscr{A} \to \mathrm{proj}(\mathscr{H})$ that is countably additive such that $E(\Omega) = 1$. For E to be countably additive means that for each sequence $(\Delta_n)_{n=1}^\infty$ of pairwise disjoint sets $\Delta_n \in \mathscr{A}$ and each $v \in \mathscr{H}$,

$$E\left(\bigcup_{n=1}^\infty \Delta_n\right) v = \sum_{k=0}^\infty E(\Delta_k)v := \lim_{n\to\infty} \sum_{k=0}^n E(\Delta_k)v,$$

where the limit is taken with respect to the strong operator topology. This implies that E is finitely additive also.

Now given a spectral measure $E : \mathscr{A} \to \mathrm{proj}(\mathscr{H})$, there is a complex measure $\mu_{vw} : \mathscr{A} \to \mathbb{C}$ for every pair $v, w \in \mathscr{H}$ given by

$$\mu_{vw} : \Delta \mapsto \langle v, E(\Delta)w \rangle.$$

THEOREM 2.5 (Spectral Theorem, cf. Thm 8.56 in Moretti's book [21] and Thm 13.30 in Rudin's book [25]). *Let $A : \mathfrak{D}(A) \to \mathscr{H}$ be a self-adjoint operator on some Hilbert space $\mathscr{H}$. Then there exists a spectral measure $E : \mathfrak{B}(\sigma(A)) \to \mathrm{proj}(\mathscr{H})$, where $\mathfrak{B}(\sigma(A))$ is the Borel σ-algebra on the spectrum $\sigma(A)$ of A, such that for all $w \in \mathfrak{D}(A)$ and $v \in \mathscr{H}$*

$$\langle v, Aw\rangle = \int_{\sigma(A)} z \; d\mu_{vw}(z).$$

This result is fundamental in quantum mechanics. Suppose that $\mathscr{H}$ is the Hilbert space of some quantum mechanical system prepared in a state $\psi \in \mathbb{S}\mathscr{H}$. Fix some observable A, which gives rise to a spectral measure E as above by the spectral theorem. Now the measure $p_{A,\psi} := \mu_{\psi\psi}$ is a probability measure (since E is projection-valued and ψ is normalized). Physically, the set $\sigma(A)$ consists of all possible measurements of the observable A, with the probability of some measurement falling within a Borel set $\Delta \subseteq \sigma(A)$ given by

$$p_{A,\psi}(\Delta) = \langle \psi, E(\Delta)\psi\rangle = \|E(\Delta)\psi\|^2.$$

Principles of quantum mechanics predict that the new state $\tilde{\psi} \in \mathbb{S}\mathscr{H}$ the system is in after such a measurement is given by

$$\tilde{\psi} = \frac{E(\Delta)\psi}{\|E(\Delta)\psi\|}.$$

This discussion shows that quantum mechanics is fundamentally probabilistic. It is therefore useful to introduce the notion of an *expected value* for the observable A. The identity 1_A on $\sigma(A)$ can be viewed as a random variable (that is a measurable function), hence we define the expectation value of A as $\langle A\rangle = \mathbb{E}(1_A)$, the expectation value of 1_A. Explicitly this is given by the formula

$$\langle A\rangle = \int_{\sigma(A)} z \; dp_{A,\psi} = \left\langle \psi, \left(\int_{\sigma(A)} z \; dE\right)\psi \right\rangle = \langle \psi, A\psi\rangle.$$

The uncertainty σ_A of this measurement is the standard deviation of 1_A given by $\sigma_A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. This leads to the **uncertainty principle** according to Heisenberg [19], Robertson [24], and Schrödinger [26]:

PROPOSITION 2.6 (Uncertainty Principle). *Let A, B be self-adjoint operators on $\mathscr{H}$. Then*

$$\sigma_A^2\sigma_B^2 \geq \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2 + \left|\frac{1}{2i}\langle[A,B]\rangle\right|^2$$

where $\{A,B\} = AB + BA$ resp. $[A,B] = AB - BA$ is the anticommutator resp. commutator of A and B.

PROOF. Firstly we have

$$\|(A - \langle A\rangle)\psi\|^2 = \langle A\psi, A\psi\rangle - 2\mathfrak{Re}\langle A\psi, \langle A\rangle\psi\rangle + \langle\langle A\rangle\psi, \langle A\psi\rangle\rangle$$

Since A is self-adjoint and $\|\psi\| = 1$, this becomes

$$\|(A - \langle A\rangle)\psi\|^2 = \left\langle \psi, A^2\psi\right\rangle - 2\mathfrak{Re}(\langle A\rangle \cdot \langle \psi, A\psi\rangle) + \langle A\rangle^2 = \left\langle A^2\right\rangle - \langle A\rangle.$$

Therefore, we have the equality $\sigma_X^2 = \|(X - \langle X\rangle)\psi\|^2$ for $X = A, B$. A nice application of the Cauchy–Schwartz inequality gives

$$\sigma_A^2\sigma_B^2 \geq |\langle(A - \langle A\rangle)\psi, (B - \langle B\rangle)\psi\rangle|^2.$$

We compute the real and imaginary parts of the inner product above. In particular,

$$\begin{aligned}\langle (A - \langle A\rangle)\psi, (B - \langle B\rangle)\psi\rangle &= \langle \psi, (A - \langle A\rangle)(B - \langle B\rangle)\psi\rangle \\ &= \langle AB\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\ &= \langle AB\rangle - \langle A\rangle\langle B\rangle.\end{aligned}$$

Similarly we have $\langle (B - \langle B\rangle)\psi, (A - \langle A\rangle)\psi\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle$. Therefore,

$$\begin{aligned}\mathfrak{Re}\langle (A - \langle A\rangle)\psi, (B - \langle B\rangle)\psi\rangle &= \frac{1}{2}\left|\langle AB\rangle - \langle A\rangle\langle B\rangle + \langle BA\rangle - \langle A\rangle\langle B\rangle\right| \\ &= \left|\frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle\right|\end{aligned}$$

$$\begin{aligned}\mathfrak{Im}\langle (A - \langle A\rangle)\psi, (B - \langle B\rangle)\psi\rangle &= \frac{1}{2i}\left|\langle AB\rangle - \langle A\rangle\langle B\rangle - \langle BA\rangle + \langle A\rangle\langle B\rangle\right| \\ &= \left|\frac{1}{2}\langle [A, B]\rangle\right|,\end{aligned}$$

completing the proof. □

We end this section with the physics of quantum mechanics. Here the Hilbert space of states is $\mathscr{H} = L^2(\mathbb{R})$ with respect to the standard Lebesgue measure. We define the **position** resp. **momentum** operators $\hat{x}$ resp. $\hat{p}$ as follows:

$$\begin{aligned}\hat{x} &: \mathfrak{D}(\hat{x}) \ni \psi \mapsto x\psi \\ \hat{p} &: \mathfrak{D}(\hat{p}) \ni \psi \mapsto -i\hbar\frac{\partial\psi}{\partial x}\end{aligned}$$

where $\hbar$ is a real number called **Planck's constant** and the respective domains are given by

$$\begin{aligned}\mathfrak{D}(\hat{x}) &= \{\psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |x\psi(x)|^2 \, dx < \infty\} \\ \mathfrak{D}(\hat{p}) &= \{\psi \in L^2(\mathbb{R}) : \exists g \in L^2(\mathbb{R}) \text{ s.t.} \\ &\qquad -\int_{\mathbb{R}} \psi(x)\frac{\partial\varphi}{\partial x} \, dx = \int_{\mathbb{R}} g(x)\varphi(x) \, dx \ \forall\varphi \in \mathscr{C}^\infty_{\text{cpt}}(\mathbb{R})\}.\end{aligned}$$

where $\mathscr{C}^\infty_{\text{cpt}}(\mathbb{R})$ are the real-valued $\mathscr{C}^\infty$-functions on $\mathbb{R}$ with compact support. We mention without proof that $\hat{x}$ and $\hat{p}$ are densely-defined, closed, and essentially self-adjoint operators on $\mathscr{H}$, since the proof of this is outside the scope of this paper. We also have the famous **canonical commutation relation** $[\hat{x}, \hat{p}] = i\hbar$, which is easily computed:

$$[\hat{x}, \hat{p}]\psi = \hat{x}(\hat{p}\psi) - \hat{p}(\hat{x}\psi) = i\hbar\left(-x\frac{\partial\psi}{\partial x} + \psi + x\frac{\partial\psi}{\partial x}\right) = i\hbar 1(\psi).$$

As in the classical case, there is a distinguished observable H, also called the **Hamiltonian** of the system. The pure states $\psi \in \mathbb{S}\mathscr{H}$ evolve via the dynamics governed by Schrödinger's equation

$$\frac{d\psi}{dt} = -iH\psi.$$

For more on quantum mechanics, consult the notes [23] (for the mathematics of quantum mechanics) as well as the books [18] and [21].

2.3. Some Words on Quantization. We've seen previously that the mathematical structures of classical and quantum mechanics contrast each other. This makes the question of quantization seem hopeless. Indeed, quantization schemes can be highly nontrivial, and sometimes what it *means* to be a "quantization" is not even well-posed. We discuss some of these ideas here, motivating the concept of deformation quantization that will be introduced in section 3.

One of the first attempts at quantization is due to the physicist P. Dirac, which we will call Dirac quantization, and is the following procedure: let W be a classical phase space and $\mathscr{H}$ some complex Hilbert space. Let $\mathfrak{S}(\mathscr{H})$ be some set of self-adjoint operators on $\mathscr{H}$. Then $\mathfrak{S}(\mathscr{H})$ has a Lie bracket given by $i/\hbar$ times the commutator, and we have seen before that $\mathscr{C}^\infty(W)$ has a Lie structure also.

DEFINITION 2.7. Let $A \subseteq \mathscr{C}^\infty(W)$ be a Poisson subalgebra and $B \subseteq \mathfrak{S}(\mathscr{H})$ a Lie subalgebra. A **Dirac quantization** is a Lie algebra morphism $\mathfrak{q} : A \to B$, in particular $\mathfrak{q}$ satisfies the following:

(1) $1 = \mathfrak{q}(1)$.
(2) $\mathfrak{q}(f)\mathfrak{q}(g) - \mathfrak{q}(g)\mathfrak{q}(f) =: [\mathfrak{q}(f), \mathfrak{q}(g)] = -i\hbar\mathfrak{q}(\{f, g\})$.

The preservation of Lie brackets is sometimes referred to as the *correspondence principle*. Although sufficient for most purposes of physics, this quantization faces some mathematical problems. A well-known result by Groenewold [15] (and later Van Hove [30]) prohibits the existence of a map $\mathfrak{q}$ with Dirac's desired propositionerties. However, by relaxing the correspondence principle to

$$[\mathfrak{q}(f), \mathfrak{q}(g)] = -i\hbar\mathfrak{q}(\{f, g\}) + O(\hbar^2) \tag{2.4}$$

so that Dirac's original requirement is recovered *asymptotically* when $\hbar \to 0$, we can avoid this no-go theorem. We give an example of such a quantization for classical *polynomial* observables due to [4], which uses what Bordemann calls *elementary star-products.*

Consider the classical phase space $W = T^*\mathbb{R} \cong \mathbb{R}^2$ with its canonical symplectic structure $\omega = dq \wedge dp$ under canonical coordinates (q, p). We choose $\mathscr{H} = L^2(\mathbb{R})$ of square Lebesgue integrable functions as the quantum phase space of states. Now the space of classical observables consist of (smooth) *real*-valued functions, but it will be convenient to first work with $\mathscr{C}^\infty(W, \mathbb{C})$ of *complex*-valued functions for now.

PROPOSITION 2.8. *The set $\mathbb{C}[q, p]$ is a Poisson subalgebra of $\mathscr{C}^\infty(W, \mathbb{C})$, i.e. $\mathbb{C}[q, p]$ is closed under the Poisson bracket.*

PROOF. The Poisson bracket on $\mathscr{C}^\infty(W, \mathbb{C})$ takes the canonical form (2.2). By bilinearity, it suffices to prove the claim for monomials $f = q^n p^m$ and $g = q^r p^s$. Indeed,

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = (ns - rm)q^{n+r-1}p^{m+s-1}$$

is yet another polynomial. $\square$

Now define an operator-valued map $\mathfrak{q}_s$ on $\mathbb{C}[q, p]$ on generators via $\mathfrak{q}_s(1) = 1$, $\mathfrak{q}_s(q) = \hat{q}$ and $\mathfrak{q}_s(p) = \hat{p}$. For the mixed monomials $q^n p^m$, there are various ways to "order" the corresponding operators $\hat{q}$ and $\hat{p}$. The tempting approach is to put $\mathfrak{q}_s(q^n p^m) = \hat{q}^n \hat{p}^m$, which is called **standard ordering**. Obviously we extend $\mathbb{C}$-linearly to all of $\mathbb{C}[q,p]$.

PROPOSITION 2.9. *The map $\mathfrak{q}_s$ is a linear isomorphism onto its image. Furthermore,*

$$\mathfrak{q}_s(\mathbb{C}[q,p]) = \left\{ \sum_{k=0}^{N} f_k \frac{d^k}{dq^k} : f_k \in \mathbb{C}[q], N \in \mathbb{Z}_{\geq 0} \right\} = (\mathbb{C}[q]) \left[\frac{d}{dq} \right].$$

PROOF. Obviously f is an injection. Now given some differential operator of the form $\sum_{k=0}^{N} f_k \frac{d^k}{dq^k}$, we have

$$\mathfrak{q}_s : \sum_{k=0}^{N} \left(\frac{f_k}{(-i\hbar)^k} p^k \right) \mapsto \sum_{k=0}^{N} f_k \frac{d^k}{dq^k}$$

which completes the proof.

□

This choice of ordering, however, is unsatisfactory in terms of Dirac quantization. For the operator $\mathfrak{q}_s(qp) = \hat{q}\hat{p}$, one has e.g. for $\psi, \varphi \in \mathscr{C}^\infty_{\mathrm{cpt}}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ that

$$\begin{aligned} \langle \hat{q}\hat{p}\psi, \varphi \rangle &= \int_{\mathbb{R}} \overline{\left(-i\hbar q \frac{\partial \psi}{\partial q} \right)} \varphi \; dq = i\hbar \int_{\mathbb{R}} (q\varphi) \frac{\partial \psi}{\partial q} \; dq \\ &= -i\hbar \int_{\mathbb{R}} \psi \left(\varphi + q \frac{\partial \varphi}{\partial q} \right) \; dq = \langle \psi, (\hat{q}\hat{p} - i\hbar)\varphi \rangle \end{aligned}$$

In other words, $\mathfrak{q}_s(qp)$ is not even a symmetric operator, hence cannot be self-adjoint in $L^2(\mathbb{R})$. The way around this is to completely symmetrize the polynomials in the operators $\hat{q}$ and $\hat{p}$. This type of ordering is called the **Weyl–Moyal** ordering.

Define a linear map $T : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ by

$$Tf = \exp\left(\frac{\hbar}{2i} \frac{\partial^2}{\partial q \partial p} \right) f = f + \frac{\hbar}{2i} \frac{\partial^2 f}{\partial q \partial p} + \frac{1}{2!} \frac{\hbar^2}{(2i)^2} \frac{\partial^4 f}{\partial q^2 \partial p^2} + \cdots \tag{2.5}$$

which is well-defined since polynomials have vanishing derivatives for sufficiently high orders. Put $\mathfrak{q}_w(f) = \mathfrak{q}_s(Tf)$ for all $f \in \mathbb{C}[q,p]$, which defines a map $\mathfrak{q}_w : \mathbb{C}[q,p] \to \mathfrak{S}(\mathscr{H})$. It turns out that $\mathfrak{q}_w$ is the total symmetrization we need, for example we have

$$\mathfrak{q}_w(qp) = \mathfrak{q}_s\left(qp + \frac{\hbar}{2i} \right) = \hat{q}\hat{p} - \frac{1}{2} i\hbar = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}).$$

Furthermore T is bijective, so that $\mathfrak{q}_w = \mathfrak{q}_s T$ is a linear isomorphism onto its image.

Eventually one looks to restrict $\mathfrak{q}_w$ to $\mathbb{R}[q,p]$, the set of real classical polynomial observables and would like this to be a Dirac quantization of $\mathbb{R}[q,p]$. To verify this, we check that $\mathfrak{q}_w$ satisfies the weakened correspondence principle (2.4). There is a clever way to do this by working directly within $\mathbb{R}[q,p]$ instead of computing $\mathfrak{q}_w(f)\mathfrak{q}_w(g) - \mathfrak{q}_w(g)\mathfrak{q}_w(f)$ for all $f, g \in \mathbb{R}[q,p]$.

PROPOSITION 2.10 (Elementary Star-Product, cf. Prop 1.2 in [4]). *Define a $\mathbb{C}$-bilinear map $\star : \mathbb{C}[q,p] \times \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ by*

$$(f,g) \mapsto f \star g = \mathfrak{q}_w^{-1}(\mathfrak{q}_w(f)\mathfrak{q}_w(g)).$$

Then $\star$ defines a noncommutative associative product on $\mathbb{C}[q,p]$. Furthermore, the following equality holds:

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{\partial^n f}{\partial q^k \partial p^{n-k}} \frac{\partial^n g}{\partial q^{n-k} \partial p^k} \tag{2.6}$$

The idea of $\star$ is to pull-back the noncommutative structure of operators on $\mathscr{H}$ onto the classical observables. We compute at lowest orders using (2.6)

$$f \star g = fg + \frac{i\hbar}{2}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right) + O(\hbar^2) = fg - \frac{i\hbar}{2}\{f,g\} + O(\hbar^2).$$

Therefore, $f \star g - g \star f = -i\hbar\{f,g\} + O(\hbar^2)$ which holds if and only if $[\mathfrak{q}_w(f), \mathfrak{q}_w(g)] = -i\hbar\mathfrak{q}_w(\{f,g\}) + O(\hbar^2)$. In other words, $\mathfrak{q}_w$ defines a quantization on the space of classical polynomial observables $\mathbb{R}[q,p]$ (in fact $\mathbb{C}[q,p]$ if complex observables are to be considered).

Remark 2.11. In principle, it is possible to extend $\mathfrak{q}_w$ to a larger class of functions, namely the Schwartz class of rapidly decreasing $\mathscr{C}^\infty$-functions. This extension is called the **Weyl quantization** and is given by [29]

$$\mathfrak{q}(f) = \int_{\mathbb{R}^2} \mathscr{F}(f)(\xi,\eta) \exp(\frac{i}{\hbar}(\hat{p}\cdot\xi + \hat{q}\cdot\eta)) d\xi d\eta,$$

where $\mathscr{F}(f)$ is the Fourier transform of f, but we only mention this here. The interested reader may refer to [11] on the Weyl quantization. All of these constructions may be generalized to $\mathbb{R}^{2n}$ analogously.

One of the properties of $\star$ is that it coincides with the original (commutative) pointwise product on $\mathscr{C}^\infty(W)$ if $\hbar = 0$. As we've seen, the linear term in $\hbar$ is the Poisson bracket on $\mathscr{C}^\infty(W)$. Thus $\star$ can be viewed as a *deformation* of the original product in the direction of the Poisson bracket. The isomorphism provided by $\mathfrak{q}_w$ allows one to further "forget" the algebra of quantum observables and work directly within $\mathscr{C}^\infty(W)$ under $\star$. This is the motivation of the deformation approach to quantization. Indeed we will see later that (2.6) is an elementary example of a deformation quantization.

3. Quantization as a Theory of Deformation

In the previous section, we described quantization as a process of translating from a classical system to a quantum mechanical system and introduced some approaches. We now move to develop the deformation approach à la Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [2]. In doing so, we introduce some deformation theory of (associative) algebras according to Gerstenhaber [14] and define deformation quantization in this language.

3.1. Deformations of Associative Algebras. For this section, let $(\mathscr{A}, \mu_0)$ always be an algebra over a commutative ring R. Here **algebra** means that $\mathscr{A}$ is an R-module with an associative multiplication $\mu_0 : (a,b) \mapsto ab$ and a unit 1. Recall that one forms the linear space $\mathscr{A}[\![\hbar]\!]$ consisting of formal power series

$$a(\hbar) = \sum_{n=0}^{\infty} \hbar^n a_n$$

with coefficients $a_n \in \mathscr{A}$, where $\hbar$ is interpreted as a formal parameter. If $\mathscr{A} = R$ is a commutative ring, then so is $R[\![\hbar]\!]$ under the product

$$a(\hbar)b(\hbar) = \sum_{n=0}^{\infty} \hbar^n \sum_{r=0}^{n} a_r b_{n-r}.$$

Motivated by the previous section, one would like a deformation of an algebraic product μ_0 to be a map of the form $\mu : \mathscr{A} \to \mathscr{A}[\![\hbar]\!]$, where $\hbar$ is now some *formal parameter*, such that $\mu = \mu_0$ when $\hbar = 0$. One also expects μ to uphold certain properties of μ_0, such as associativity (or some form of it) and preservation of units. This leads us to the following definition:

Definition 3.1. A **(formal associative) deformation** of $(\mathscr{A}, \mu_0)$ is a sequence $(\mu_n)_{n=0}^{\infty}$ of R-bilinear maps $\mu_n : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that for all $x, y, z \in \mathscr{A}$,

(1) $\sum_{s=0}^{r}(\mu_r(\mu_{r-s}(x,y),z) - \mu_r(x, \mu_{r-s}(y,z))) = 0$ for all $r \geq 0$.
(2) $\mu_r(1,x) = 0 = \mu_r(x,1)$ for all $r \geq 1$.

Note that the zeroth term of the sequence $(\mu_n)_{n=0}^{\infty}$ is intended to be the product μ_0 on $\mathscr{A}$. Given a such a deformation, there is an induced R-bilinear map $\mu_\hbar : \mathscr{A} \times \mathscr{A} \to \mathscr{A}[\![\hbar]\!]$ given by the formal series

$$\mu_\hbar(x,y) = \sum_{n=0}^{\infty} \hbar^n \mu_n(x,y) = xy + \hbar\mu_1(x,y) + O(\hbar^2)$$

called the **deformed product** of $(\mathscr{A}, \mu_0)$. Thus one is justified to say "let $\mu_\hbar$ be a formal deformation of $\mathscr{A}$," understanding the associated sequence of maps in Definition 3.1. Furthermore if the deformation parameter is understood, it may be omitted from the subscript so that one writes μ for $\mu_\hbar$. We show now that μ is the "expected" definition for a deformation.

Proposition 3.2 (cf. Prop 2.1 in [4]). *A deformed product μ extends to an $R[\![\hbar]\!]$-bilinear map $\mathscr{A}[\![\hbar]\!] \times \mathscr{A}[\![\hbar]\!] \to \mathscr{A}[\![\hbar]\!]$, written by μ also, given by*

$$\mu(a(\hbar), b(\hbar)) = \sum_{n=0}^{\infty} \hbar^n \sum_{r+s+t=n} \mu_r(a_s, b_t).$$

Furthermore, μ forms $\mathscr{A}[\![\hbar]\!]$ into an algebra over $R[\![\hbar]\!]$ with unit 1.

Proof. It suffices to check that μ is associative when restricted to $\mathscr{A}$ thanks to bilinearity. Indeed,

$$\begin{aligned}
\mu\big(\mu(a,b),c\big) &= \mu\left(\sum_{n=0}^{\infty} \hbar^n \mu_n(a,b)\right) = \sum_{n=0}^{\infty} \hbar^n \mu\big(\mu_n(a,b),c\big) \\
&= \sum_{n=0}^{\infty} \hbar^n \left(\sum_{m=0}^{\infty} \hbar^m \mu_m\big(\mu_n(a,b),c\big)\right) = \sum_{r=0}^{\infty} \hbar^r \sum_{n+m=r} \mu_m\big(\mu_n(a,b),c\big) \\
&= \sum_{r=0}^{\infty} \hbar^r \sum_{m=0}^{r} \mu_m\big(\mu_{r-m}(a,b),c\big) = \sum_{r=0}^{\infty} \hbar^r \sum_{m=0}^{r} \mu_m\big(a, \mu_{r-m}(b,c)\big) \\
&= \sum_{n=0}^{\infty} \hbar^n \left(\sum_{m=0}^{\infty} \hbar^m \mu_m\big(a, \mu_n(b,c)\big)\right) = \sum_{n=0}^{\infty} \hbar^n \mu\big(a, \mu_n(b,c)\big) \\
&= \mu\big(a, \mu(b,c)\big)
\end{aligned}$$

where we have used condition (1) of definition 3.1. Therefore, μ is an associative product on $\mathscr{A}[\![\hbar]\!]$. Now condition (2) implies

$$\mu(1, a(\hbar)) = 1 \cdot a(\hbar) + \sum_{n=1}^{\infty} \hbar^n \mu_n(1, a(\hbar)) = a(\hbar)$$

so that 1 remains a unit in $\mathscr{A}[\![\hbar]\!]$.

□

REMARK 3.3. In this language, the product $\star$ of proposition 2.10 is a deformed product of $(\mathscr{C}^\infty(W), \cdot)$ with deformation parameter $i\hbar/2$.

Obviously setting $\mu_n = 0$ for all $n \geq 1$ constitutes a formal deformation of any algebra, which we will call a **trivial formal deformation.** This is hardly a deformation at all, so a more interesting question to consider is when an algebra admits a *nontrivial* formal deformation. Cohomological methods are used to construct such maps μ_n, so the answer to this question lies here.

We recall that an algebra $(\mathscr{A}, \mu_0)$ gives rise to the **Hochschild cochain complex** $(C^\bullet(\mathscr{A}), \beta)$, whose n-cochains are the n-multilinear maps $\mathscr{A}^n \to \mathscr{A}$ and whose coboundary map $\beta : C^\bullet(\mathscr{A}) \to C^{\bullet+1}(\mathscr{A})$ is given by

$$\begin{aligned}\beta\eta(x_1, \ldots, x_{n+1}) = x_1\eta(x_2, \ldots, x_{n+1}) \\ + \sum_{r=1}^{n} (-1)^r \eta(x_1, \ldots, x_{r-1}, x_r x_{r+1}, \ldots, x_{n+1}) \\ + (-1)^{n+1}\eta(x_1, \ldots, x_n)x_{n+1},\end{aligned}$$

for each $\eta \in C^n(\mathscr{A})$, $n \geq 0$. The associated cohomology theory, denoted by $HH^\bullet(\mathscr{A})$, is called the **Hochschild cohomology** of $\mathscr{A}$.

The Hochschild cochain complex has the structure of a differential graded Lie algebra (briefly DGLA) [17]. We declare elements of $C^n(\mathscr{A})$ to have degree $n-1$, i.e. if $\eta \in C^n(\mathscr{A})$ then $|\eta| = n - 1$. Define the element $\omega \smile \eta \in C^{|\omega|+|\eta|+1}(\mathscr{A})$ by

$$\begin{aligned}\omega \smile \eta(x_1, \ldots, x_{|\omega|+|\eta|+1}) = \sum_{k=1}^{|\omega|+1} (-1)^{(k-1)|\eta|} \times \\ \omega(x_1, \ldots, x_{k-1}, \eta(x_k, \ldots, x_{k+|\eta|}), \ldots, x_{|\omega|+|\eta|+1}).\end{aligned}$$

The **Gerstenhaber bracket** $[\omega, \eta]_G$ of ω and η is then the degree $|\omega| + |\eta| + 1$ element given by

$$[\omega, \eta]_G = \omega \smile \eta - (-1)^{|\omega||\eta|}\eta \smile \omega,$$

which is a super Lie bracket on $C^\bullet(\mathscr{A})$. The coboundary map is related to the Gerstenhaber bracket in the sense that $\beta\eta = [\mu_0, \eta]_G$, a discussion of which can be found in [12]. We mention that a bilinear map $m \in C^2(\mathscr{A})$ defines an associative product on $\mathscr{A}$ if and only if $[m, m]_G = 0$, which is easily seen by computing:

$$[m, m]_G(x, y, z) = m(m(x, y), z) - m(x, m(y, z)) + m(m(x, y), z) - m(x, m(y, z)).$$

In general, we have the following:

LEMMA 3.4 (cf. Exercise 2.17 in [17]). *Let $(\mu_n)_{n=0}^{\infty}$ be a deformation of $(\mathscr{A}, \mu_0)$. Then the associativity condition (1) of definition 3.1 is equivalent to*

$$\beta\mu_r(x, y, z) = \sum_{\substack{s+t=r \\ s,t>0}} (\mu_s(\mu_t(x, y), z) - \mu_s(x, \mu_t(y, z)), \tag{3.1}$$

for each $r \geq 1$ (the sum is interpreted as zero if $r = 1$).

PROOF. For any $r \geq 0$ we have $\beta\mu_r(x,y,z) = x\mu_r(y,z) - \mu_r(xy,z) + \mu_r(x,yz) - \mu_r(x,y)z$. The associativity condition reads $\partial\mu_1 = 0$ and

$$
\begin{aligned}
0 &= \sum_{s+t=r} \Big(\mu_s(\mu_t(x,y),z) - \mu_s(x,\mu_t(y,z)) \Big) \\
&= \mu_r(x,y)z - \mu_r(x,yz) + \sum_{\substack{s+t=r \\ s,t>0}} \Big(\mu_s(\mu_t(x,y),z) - \mu_s(x,\mu_t(y,z) \Big) \\
&\quad + \mu_r(xy,z) - x\mu_r(y,z) \\
&= -\beta\mu_r(x,y,z) + \sum_{\substack{s+t=r \\ s,t>0}} (\mu_s(\mu_t(x,y),z) - \mu_s(x,\mu_t(y,z)),
\end{aligned}
$$

for $r > 1$, which proves equation (3.1).

$\square$

Lemma 3.4 and the discussion above roughly says that Hochschild cohomology can be viewed as a measure of how "associative" deformations are. This hints at how to proceed with the existence of nontrivial deformations. Suppose that μ_1 is a given Hochschild 2-cocycle. Then the map $\mu = \mu_0 + \hbar\mu_1$ defines a formal deformation up to linear order. We would like to extend μ to be a deformation to second order.

Denote by C_r the 2-cochain on the RHS of Equation (3.1). Then $C_2(x,y,z) = \mu_1(\mu_1(x,y),z) - \mu_1(x,\mu_1(y,z)) = \frac{1}{2}[\mu_1,\mu_1]_G(x,y,z)$, and in particular

$$\beta[\mu_1,\mu_1]_G = [\mu_0,[\mu_1,\mu_1]_G]_G = -2[\mu_1,\partial\mu_1]_G = 0$$

by the (super) Jacobi identity. Thus C_2 determines an element of $HH^3(\mathscr{A})$. If this element is zero, then there exists some Hochschild 2-cochain μ_2 with $C_2 = \beta\mu_2$, which by Lemma 3.4 turns the extended map $\mu = \mu_0 + \hbar\mu_1 + \hbar^2\mu_2$ into a formal deformation of order 2. In general, given a formal deformation of order $n-1$ one can show that C_n is a 2-cocycle. If $C_n = \beta\mu_n$ is furthermore a coboundary, then the deformation can be extended to order n [14]. In other words, we have the following:

THEOREM 3.5 (Obstructions to Deformations, cf. Prop 2.2 in [12]). *The obstructions for formal deformations of an algebra $\mathscr{A}$ lie in $HH^3(\mathscr{A})$. In particular, if $HH^3(\mathscr{A}) = 0$, then any formal deformation of any order can be arbitrarily extended.*

Some authors call $HH^3(\mathscr{A})$ the *obstruction space* of $\mathscr{A}$. It is important to note that theorem 3.5 is not a necessary condition, i.e. it is possible to extend deformations when $HH^3(\mathscr{A})$ is nontrivial. Nevertheless, Hochschild cohomology still gives a method of constructing deformations.

Given any two deformations, there is a notion of *morphisms* between them.

DEFINITION 3.6. Let μ and $\tilde{\mu}$ be two deformations of an algebra $\mathscr{A}$. A **morphism of formal deformations** from μ to $\tilde{\mu}$ is a formal power series $T = \sum_{n=0}^{\infty} \hbar^n T_n$ of algebra endomorphisms $T_n : \mathscr{A} \to \mathscr{A}$ such that $T\mu = \tilde{\mu}T^2$.

If T has a (formal) inverse, then T is said to be an **isomorphism of formal deformations** and one says that μ and $\tilde{\mu}$ are **c-equivalent** deformations. The "c" stands for "cohomologically" [2]. The following gives a formula for such an inverse:

PROPOSITION 3.7. *Let $T = \sum_{n=0}^{\infty} \hbar^n T_n$ be a morphism of deformations μ and $\tilde{\mu}$. If $T_0 = 1$, then T has a formal inverse $T^{-1} = \sum_{n=0}^{\infty} \hbar^n \tilde{T}_n$, where the $\tilde{T}_n$ are recursively determined by the formula*

$$\tilde{T}_n = \begin{cases} 1 & \text{if } n = 0. \\ -\displaystyle\sum_{s=1}^{n} \tilde{T}_{n-s} T_s & \text{if } n > 0. \end{cases}$$

PROOF. For any $x \in \mathscr{A}$, we compute

$$x = T^{-1}(Tx) = \left(1 + \sum_{n=1}^{\infty} \hbar^n \tilde{T}_n\right)\left(x + \sum_{m=1}^{\infty} \hbar^m T_m x\right) = x + \sum_{n=1}^{\infty} \hbar^n \left(\sum_{s=0}^{n} \tilde{T}_{n-s} T_s x\right)$$

which gives the desired result.

□

This covers the basic elements of Gerstenhaber's deformation theory of algebras. We refer the reader to [13] for more details not used in this paper.

3.2. Quantization as a Formal Deformation: Star-Products. We proceed with the plan to describe the quantization of a classical system M as a deformation of its associative algebra $\mathscr{C}^\infty(M)$ of classical observables. The following definition is equivalent to the original one presented in [2]:

DEFINITION 3.8. Let (M, π) be a Poisson manifold. A **deformation quantization** of M is a formal associative deformation $(\mu_n)_{n=0}^{\infty}$ of $\mathscr{C}^\infty(M)$ such that the μ_n are bi-differential operators for $n \geq 1$ satisfying the so-called *classical limit*:

$$\mu_1(f, g) - \mu_1(g, f) = i\pi(df, dg) \quad \text{for all } f, g \in \mathscr{C}^\infty(M).$$

The deformed product of a deformation quantization is written $\star$. It is called a **(differential) star-product** on M, the "differential" referring to the condition that the μ_n are *bi-differential operators*. This means that in a chart (U, x^i) of M, the μ_n take the form

$$\mu_n(u, v)|_U = \sum_{\substack{|I| \leq N_I \\ |J| \leq N_J}} D^{I,J} \partial_I(u|_U) \partial_J(v|_U)$$

for $\dim M$-tuples I and J with $D^{I,J} \in \mathscr{C}^\infty(M)$. Multi-index notation is used, e.g. if $I = (i_1, \ldots, i_m)$ then ∂_I stands for $\frac{\partial^{i_1 + \cdots + i_m}}{\partial x_{i_1} \cdots \partial x_{i_m}}$ and $|I| = i_1 + \cdots + i_m$.

Previously we mentioned that Hochschild cohomology governs the construction of arbitrary associative deformations. Here, we are interested in local (differential) deformations of $\mathscr{C}^\infty(M)$. There exists a subcomplex $(C^\bullet_{\text{diff}}(\mathscr{C}^\infty(M)), \beta)$ of the Hochschild complex whose cochains are the n-differential operators on $\mathscr{C}^\infty(M)$ called the **differential Hochschild complex.** The associated cohomology theory is called **differential Hochschild cohomology,** denoted by $HH^\bullet_{\text{diff}}(\mathscr{C}^\infty(M))$. It turns out that Theorem 3.5 restricts to the differential complex, i.e. the obstructions to formal *differential* deformations of $\mathscr{C}^\infty(M)$ lie in $HH^3_{\text{diff}}(\mathscr{C}^\infty(M))$ [16].

One can also define a so-called *star-commutator* by

$$[f,g]_\star = \frac{1}{i\hbar}\big(f\star g - g\star f\big).$$

The classical limit says that the star-commutator is a deformation of the Poisson bracket [13], analagous to $\star$ being a deformation of the pointwise product. Indeed,

$$\begin{aligned}[f,g]_\star &= \frac{1}{i\hbar}\left(\sum_{n=0}^{\infty}\hbar^n\mu_n(f,g) - \sum_{n=0}^{\infty}\hbar^n\mu_n(g,f)\right)\\ &= \frac{1}{i\hbar}\left(fg - gf + \hbar\big(\mu_1(f,g)-\mu_1(g,f)\big) + O(\hbar^2)\right)\\ &= \{f,g\} + O(\hbar).\end{aligned}$$

One can, however, go one step further. It turns out that any star-product is c-equivalent to one whose linear term defines a Poisson bracket on $\mathscr{C}^\infty(M)$. We show this now with a series of lemmas.

LEMMA 3.9 (cf. Lemma 1.10 in [12]). *Let $\mu_1 \in C^2(\mathscr{C}^\infty(M))$ be a 2-cocycle. If μ_1 is skew-symmetric then μ_1 satisfies the Leibniz identity.*

PROOF. Since $\beta\mu_1 = 0$, we have $\beta\mu_1(f,g,h) - \beta\mu_1(g,h,f) = 0$ and $\beta\mu_1(h,f,g) - \beta\mu_1(g,h,f) = 0$. If we assume that μ_1 is skew-symmetric, this gives

$$\begin{aligned}\mu_1(fg,h) &= 2f\mu_1(g,h) - \mu_1(f,g)h - g\mu_1(h,f) - \mu_1(g,hf)\\ \mu_1(fg,h) &= 2g\mu_1(f,h) + h\mu_1(f,g) + f\mu_1(g,h) - \mu_1(f,gh)\end{aligned}$$

Equivalently, adding the above together,

$$\begin{aligned}2\mu_1(fg,h) &= 2\big(\mu_1(f,h)g + f\mu_1(g,h)\big) - f\mu_1(h,g) + \mu_1(hf,g) - \mu_1(f,gh) + \mu_1(f,h)g\\ &= 2\big(\mu_1(f,h)g + f\mu_1(g,h)\big) - \beta\mu_1(f,h,g)\\ &= 2\big(\mu_1(f,h)g + f\mu_1(g,h)\big).\end{aligned}$$

Thus μ_1 satisfies the Leibniz identity, proving the claim.

□

LEMMA 3.10 (cf. Lemma 1.10 in [12]). *Let $\star = \sum_{n=0}^\infty \hbar^n\mu_n$ be any star-product on M. Then the bracket $\{f,g\} = \mu_1(f,g) - \mu_1(g,f)$ is a Lie bracket on $\mathscr{C}^\infty(M)$ [4].*

PROOF. Bilinearity and skew-symmetry are immediate. It remains to check the Jacobi identity. Let

$$J(f,g,h) = \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}.$$

Then $J(f,g,h)$ is a sum over cyclic permutations of the expression

$$\{\{f,g\},h\} = \mu_1\big(\mu(f,g),h\big) - \mu_1\big(\mu(g,f),h\big) - \mu_1\big(h,\mu(f,g)\big) + \mu_1\big(h,\mu(g,f)\big).$$

By lemma 3.4 one checks that this implies

$$\begin{aligned}J(f,g,h) &= (\partial\mu_2)(f,g,h) + (\partial\mu_2)(h,f,g) + (\partial\mu_2)(g,h,f)\\ &\quad - (\partial\mu_2)(g,f,h) - (\partial\mu_2)(h,g,f) - (\partial\mu_2)(f,h,g).\end{aligned}$$

A straightforward computation shows that the RHS of the above vanishes.

□

We can now prove the claim:

Theorem 3.11 (cf. Thm 1.6 in [12]). *Let $\tilde{\star} = \sum_{n=0}^{\infty} \hbar^n \tilde{\mu}_n$ be a star-product on M. Then $\tilde{\star}$ is c-equivalent to a star-product $\star = \sum_{n=0}^{\infty} \hbar^n \mu_n$ with μ_1 a Poisson bracket on $\mathscr{C}^\infty(M)$.*

Proof. Firstly, decompose $\tilde{\mu}_1$ into its (skew-)symmetric parts

$$\tilde{\mu}_1(f,g) = \tilde{\mu}_1^-(f,g) + \tilde{\mu}_1^+(f,g),$$

where $\tilde{\mu}_1^{\pm}(f,g) = \frac{1}{2}(\tilde{\mu}_1(f,g) \pm \tilde{\mu}_1(g,f))$. Now since $\tilde{\mu}_1^+$ is symmetric (and a cocycle since $\tilde{\mu}_1$ is), it determines the same element in $HH^2_{\text{diff}}(\mathscr{C}^\infty(M))$ as 0, hence there exists some $T_1 \in C^1_{\text{diff}}(\mathscr{C}^\infty(M))$ such that $\tilde{\mu}_1^+ = \partial T_1$. We form a morphism of star-products $T = \sum_{n=0}^{\infty} \hbar^n T_n$, where $T_n = 1$ whenever $n \neq 1$. This defines a new star-product by

$$f \star g = T^{-1}\big((Tf)\tilde{\star}(Tg)\big),$$

where T^{-1} is given by proposition 3.7. Expanding everything to first-order in $\hbar$ gives

$$\begin{aligned} f \star g &= \big(1 - \hbar T_1 + O(\hbar^2)\big)\big(f \tilde{\star} g + \hbar(f \tilde{\star} T_1(g) + T_1(f)\tilde{\star} g) + O(\hbar^2)\big) \\ &= fg + \hbar\big(\tilde{\mu}_1(f,g) + T_1(fg) - fT_1(g) - T_1(f)g\big) + O(\hbar^2) \\ &= fg + \hbar\big(\tilde{\mu}_1^-(f,g) + \tilde{\mu}_1^+(f,g) - (\partial T_1)(f,g)\big) + O(\hbar^2) \\ &= fg + \hbar\tilde{\mu}_1^-(f,g) + O(\hbar^2). \end{aligned}$$

In particular, we have $\mu_1 = \tilde{\mu}_1^-$. By Lemma 3.9 and Lemma 3.10, μ_1 defines a Poisson bracket on $\mathscr{C}^\infty(M)$, proving the theorem.

□

In other words, a star-product $\star$ on M can be assumed to have the form $f \star g = fg + \hbar\{f,g\} + O(\hbar^2)$ after an isomorphism of star-products. It has been shown in [2] that c-equivalence is related to operator ordering in quantum field theory, which will prove itself useful in section 4. For the time being, we end this section with some examples.

Example 3.12 (Elementary Star-Products.). Recall the product $\star$ on $\mathbb{C}[q,p]$ given by (2.6), which we write below for convenience:

$$f \star_M g = \sum_{n=0}^{\infty} \frac{(i\hbar/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{\partial^n f}{\partial q^k \partial p^{n-k}} \frac{\partial^n g}{\partial q^{n-k} \partial p^k} = fg + \frac{i\hbar}{2}\{f,g\} + O(\hbar^2).$$

Recall also the quantization map $\mathfrak{q}_s$ on $\mathbb{C}[q,p]$ given by standard ordering. If we put $f \star_S g = \mathfrak{q}_s^{-1}\big(\mathfrak{q}_s(f)\mathfrak{q}_s(g)\big)$, we get another star-product on $\mathbb{C}[q,p]$ given by [4]

$$f \star_S g = \sum_{n=0}^{\infty} \frac{(\hbar/i)^n}{n!} \frac{\partial^n f}{\partial p^n} \frac{\partial^n g}{\partial q^n}$$

called the **standard ordered star-product.** In the language developed, the map $T = \exp\left(\frac{\hbar}{2i} \frac{\partial^2}{\partial q \partial p}\right)$ before defines an isomorphism between the standard ordered and the Moyal–Weyl ordered star-products.

Example 3.13 (Moyal Product). The next stage in simplicity generalizes the star-product given by Moyal–Weyl ordering [12]. We work in a finite-dimensional real vector space V of dimension m. There is a *constant Poisson structure* on V given by any antisymmetric matrix $(\pi^{ij})_{1\leq i,j\leq m}$, i.e.

$$\pi = \frac{1}{2}\sum_{i,j=1}^{m} \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j},$$

where the (x^i) are canonical coordinates on V. Then the **Moyal product** on V is given by

$$f\star g = \exp\left(\frac{i\hbar}{2}\pi\right)(f,g) = fg + \sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n \sum_{\substack{i_1,\ldots,i_n=1\\ j_1,\ldots,j_n=1}}^{m} \pi^{i_1j_1}\cdots\pi^{i_nj_n}\left(\partial_{i_1}\cdots\partial_{i_n}f\right)\left(\partial_{j_1}\cdots\partial_{j_n}g\right)$$

Of course, all the terms are bi-differential operators and $\star$ satisfies the classical limit, hence forms a star-product on V. In the case that $m=2$ (or more generally $2k$), one may consider the antisymmetric matrix

$$\pi^{ij} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Then the corresponding Moyal product coincides with the elementary Moyal–Weyl ordered star-product from before.

4. Deformation Quantization of Fields

In this final section we investigate the ongoing research in the application of deformation quantization to field theory. The purpose of this section is to outline some key ideas in addition to discussing some challenges. Our focus will be on G. Dito's star-quantization appearing in [10].

4.1. Quantum Field Theory (QFT). One of the main challenges in QFT is in the various approaches, each depending on the physical situation at hand. The approach most relevant to the deformation programme is *perturbative field theory,* which is roughly a functional approach [22]. Typically one considers a space of fields consisting of solutions to the equation $(\Box + m^2)\Phi = 0$, where $\Box = \Delta - \partial_t^2$ is the d'Alembertian [9]. Then a phase space $\mathscr{M}\subseteq E_\infty := \mathscr{S}(\mathbb{R}^3,\mathbb{R})\oplus\mathscr{S}(\mathbb{R}^3,\mathbb{R})$ of initial data (φ,π), where $\varphi(x)=\Phi(x,0)$ and $\pi(x) = \frac{\partial\Phi}{\partial t}(x,0)$, is formed. The analogous space of observables here are then the set of scalar-valued functionals $\Psi:\mathscr{M}\to\mathbb{R}$. Then $\mathscr{M}$ has a Poisson structure given by [27] as

$$\pi(\Psi_1,\Psi_2) = \int_{\mathscr{M}}\left(\frac{\delta\Psi_1}{\delta\varphi}\frac{\delta\Psi_2}{\delta\pi} - \frac{\delta\Psi_1}{\delta\pi}\frac{\delta\Psi_2}{\delta\varphi}\right),$$

where the δ denotes a functional (Fréchet) derivative.

In a naive application of deformation quantization to fields, one may consider replacing the family of cochains $(\mu_n)_{n=1}^{\infty}$ in Definition 3.8 by a family of cochains on functionals. The hope is then to form the "star-product"

$$\Psi_1\star\Psi_2 = \Psi_1\Psi_2 + \sum_{n=1}^{\infty}\hbar^n\mu_n(\Psi_1,\Psi_2) \tag{4.1}$$

as before. The challenge here is giving meaning to this star-product, in particular having well-defined cochains μ_n on a space of functionals [9]. Physicists have learned to deal with e.g. divergences through perturbative renormalization methods, and the approach here is similar, taking advantage of the cohomology in deformation theory. Writing (4.1) down formally, one can replace diverging cochains via a c-equivalence operator $T = \sum_{n=0}^{\infty} \hbar T_n$. This is done in a manner identical to theorem 3.11, in which the linear term cochain was replaced by a Poisson bracket. In fact, one is justified to call this method *cohomological renormalization* [7].

Cohomological renormalization is the main idea in G. Dito's deformation quantization of a free scalar field. Before this investigation, we take an excursion to spectral theory in the star-product context.

4.2. Star-Exponentials and Spectral Theory. In physical applications, one looks to find e.g. energy levels of a given quantum mechanical system. This is straightforward in the standard formulation of quantum mechanics: the energy levels are given by the spectrum of an operator on a Hilbert space called the Hamiltonian. However, in the deformation setting, there may be no Hilbert spaces on which we can write down a spectral theory. This brings the need for an *autonomous* analogue of a spectral theory defined in terms of star-products on classical phase spaces. Indeed, such a theory is achieved via *star-exponentials* [3].

Definition 4.1. Fix a star-product $\star$ on a classical phase space W and let $f \in \mathscr{C}^\infty(W)$ be a classical observable. Let $t \in \mathbb{R}$. The **star-exponential** of f, denoted by $\operatorname{Exp}_\star(ft)$, is the formal power series

$$\operatorname{Exp}_\star(ft) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{i\hbar}\right)^n f^{\star n}, \qquad \text{where } f^{\star n} = \underbrace{f \star \cdots \star f}_{n\text{-times.}}.$$

The star-exponential can be viewed as a power series in t and, in particular, can be considered as a distribution on W. In this case one applies a Fourier–Dirichlet expansion

$$\operatorname{Exp}_\star(ft) = \sum_{\lambda \in I} \pi_\lambda e^{\lambda t / i\hbar},$$

where $\pi_\lambda \in \mathscr{C}^\infty(W)$ with I a sequence of complex numbers. It turns out that in the case where $\star$ is the Moyal product, then I resp. π_λ is the spectrum resp. projectors of the Weyl quantized operator $\mathfrak{q}_w(f)$ [3]. From this notion the general classical spectral theory is abstracted:

Definition 4.2. Let $f \in C(W)$ be such that $\operatorname{Exp}_\star(ft)$ has a well-defined Fourier–Dirichlet expansion. Then I is called the **spectrum** of f, the $\lambda \in I$ are called **eigenvalues** of f, and the π_λ is called the **projector associated with λ.** In this case we say that f has a **classical spectral theory.**

The following justifies the definition made:

Proposition 4.3 (cf. equations (4-3) and (4-4) in [3]). *The star-exponential satisfies the equation*

$$f \star \operatorname{Exp}_\star(ft) = i\hbar \frac{d}{dt} \operatorname{Exp}_\star(ft).$$

Furthermore, if f has a classical spectral theory then $f \star \pi_\lambda = \lambda \pi_\lambda$ and there is a spectral decomposition $f = \sum_{\lambda \in I} \lambda \pi_\lambda$.

PROOF. We compute

$$
\begin{aligned}
i\hbar\frac{d}{dt}\mathrm{Exp}_\star(ft) &= i\hbar\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(i\hbar)^n}t^{n-1}f^{\star n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{i\hbar}\right)^{n-1}f^{\star n} \\
&= f\star\sum_{n=1}\frac{1}{n!}\left(\frac{t}{i\hbar}\right)^{n-1}f^{\star(n-1)} \\
&= f\star\mathrm{Exp}_\star(ft).
\end{aligned}
$$

Now if f has a classical spectral theory, then this implies

$$
f\star\sum_{\lambda\in I}\pi_\lambda e^{\lambda t/i\hbar} = i\hbar\frac{d}{dt}\sum_{\lambda\in I}\pi_\lambda e^{\lambda t/i\hbar} = \sum_{\lambda\in I}\lambda\pi_\lambda e^{\lambda t/i\hbar},
$$

and in particular $f\star\pi_\lambda = \lambda\pi_\lambda$ for each $\lambda\in I$.

□

In physical applications, one considers the star-exponential of a classical Hamiltonian. The associated spectrum consists of the energy levels after quantization.

4.3. Star-Quantization of a Free Scalar Field. We outline the main ideas presented in [10], mostly without proof. Some computations will be shown and we present some commentary here. Star-quantization of fields is still an area of active research, and we postpone discussion of technical difficulties to the next section in order to focus on the star-quantization techniques.

Let Φ be a classical free massive scalar (real) field with initial conditions (φ,π) in Schwartz space $\mathscr{S}$. It is convenient to decompose (φ,π) into Fourier modes (a^*,a) by

$$
\begin{aligned}
\varphi(x) &= \int\left(\frac{a^*(k)e^{-i\langle k,x\rangle}+a(k)e^{i\langle k,x\rangle}}{2(2\pi)^{3/2}\omega_k}\right)dk \\
\pi(x) &= \int\left(\frac{a^*(k)e^{-i\langle k,x\rangle}+a(k)e^{i\langle k,x\rangle}}{2(2\pi)^{3/2}}\right)dk
\end{aligned}
$$

where $\omega_k = \left(\|k\|^2+m^2\right)^{1/2}$ [7]. The free scalar Hamiltonian associated to this system is given by

$$
H = \int(\omega_k a^*(k)a(k))\,dk. \tag{4.2}
$$

We follow the conventions in [10], in particular $a_1(k)=a(k)$, $a_2(k)=a^*(k)$ and

$$
\langle\delta_{i_1\cdots i_n}\Psi_1,\delta_{j_1\cdots j_n}\Psi_2\rangle := \int\left(\frac{\delta^n\Psi_1}{\delta a_{i_1}(k_1)\cdots\delta a_{i_n}(k_n)}\frac{\delta^n\Psi_2}{\delta(a_{j_1})^*(k_1)\cdots\delta(a_{j_n})^*(k_n)}\right)dk,
$$

where dk represents $dk_1\cdots dk_n$. If we take as Λ^{ij} the antisymmetric matrix in the elementary Moyal–Weyl ordered star-product before, that is

$$
\Lambda^{ij} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},
$$

then Segal's [27] Poisson structure is written as

$$\{\Psi_1, \Psi_2\} = \sum_{i,j=1}^{2} \Lambda^{ij} \langle \delta_i \Psi_1, \delta_j \Psi_2 \rangle.$$

One can attempt to define an analogous Moyal product on a certain subspace of functionals using the same formula:

$$\Psi_1 \star_M \Psi_2 := \exp\left(\frac{i\hbar}{2}\Lambda\right)(\Psi_1, \Psi_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \Lambda^{i_1 j_1} \cdots \Lambda^{i_n j_n} \langle \delta_{i_1 \cdots i_n} \Psi_1, \delta_{j_1 \cdots j_n} \Psi_2 \rangle.$$

It is known to physicists that Moyal–Weyl ordering is not suitable in field theory contexts. If one attempts to write down a spectral theory for the free scalar Hamiltonian (4.2) using star-exponential methods, one finds that $H \star H$ is already not well-defined [10]. As we have seen, the Moyal product corresponds to Moyal–Weyl ordering. Thus this classical fact is reflected here in the star-product context.

In the usual quantum field theory setting, normal ordering removes these divergences. G. Dito's approach follows this standard one, but uses cohomological renormalization instead. The "correct" morphism of star-products is given by [10]

$$T = \exp\left(\frac{\hbar}{2} \int \frac{\delta^2}{\delta a(k) \delta a^*(k)} \, dk\right)$$

which defines the normal ordered star-product

$$\Psi_1 \star_N \Psi_2 = T^{-1}\big(T\Psi_1 \star_M T\Psi_2\big) \tag{4.3}$$

Now there is a unique Gaussian measure μ on $\mathscr{S}' \oplus \mathscr{S}'$ given by the characteristic function $\exp(-\frac{1}{\hbar}\int a^*(k)a(k)\, dk)$. The star-product (4.3) has the explicit form [9]:

$$(\Psi_1 \star_N \Psi_2)(a^*, a) = \int_{\mathscr{S}' \oplus \mathscr{S}'} \Big(\Psi_1(a^*, a + \xi)\Psi_2(a^* + \xi^*, a)\Big) \, d\mu(\xi^*, \xi).$$

There is a series expansion $\Psi_1 \star_N \Psi_2 = \sum_{n=0}^{\infty} \big(\frac{i\hbar}{2}\big)^n \mu_n(\Psi_1, \Psi_2)$ with the cochains given by [10]

$$\mu_n(\Psi_1, \Psi_2) = (-2i)^n \int \left(\frac{\delta^n \Psi_1}{\delta a(k_1) \cdots \delta a(k_n)} \frac{\delta^n \Psi_2}{\delta a^*(k_1) \cdots \delta a^*(k_n)} \right) dk \tag{4.4}$$

The spectral theory of (4.2) can be written down using this normal ordered star-product. To do this, we formally compute $H \star_N \Psi$ for some functional Ψ. We have $\mu_0(H, \Psi) = H\Psi$ and

$$\mu_1(H, \Psi) = -2i \int \left(\frac{\delta H}{\delta a(k)} \frac{\delta \Psi}{\delta a^*(k)} \right) dk = -2i \int \left(\omega_k a^*(k) \frac{\delta \Psi}{a^*(k)} \right) dk.$$

Now since $\mu_n(H, \cdot) = 0$ for $n \geq 2$, the series expansion becomes

$$H \star_N \Psi = H\Psi + \hbar \int \left(\omega_k a^*(k) \frac{\delta \Psi}{\delta a^*(k)} \right) dk \tag{4.5}$$

If we denote by $\mathscr{O}$ the set of functionals Ψ for which $\frac{\delta^r \Psi(a^*, a)}{\delta a^*(k_1) \cdots \delta a^*(k_r)} \in \mathscr{S}^r$ for each $r \geq 1$, then one can show by induction that any n^{th} star-power of H will be in $\mathscr{O}$ [10]. In other words, $\mathrm{Exp}_\star(tH/i\hbar)$ exists on $\mathscr{S}$ and one proceeds using the theory described in 4.2.

4.4. Closing Remarks on Star-Quantization of Fields. The fundamental challenge in the application of deformation quantization to fields is the change from finite-dimensional to infinite-dimensional manifolds. For example, the question of existence for star-products has been solved in the finite-dimensional case; de Wilde and Lecomte gave a proof for symplectic manifolds [6], while Kontsevich proved the (very difficult) case for Poisson manifolds [20]. The existence of star-products in the infinite-dimensional case, however, faces many challenges. One cannot even expect a star-product to be defined in general on all smooth functions, just as seen in the scalar field case. Since G. Dito's star-product approach to QFT [10], progress has been made in making sense of deformation quantization on infinite-dimensional Hilbert spaces: see e.g. [8].

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