University of Colorado **Boulder**

Department of Mathematics

Summer Research In Mathematics

Derived Poisson Structure on $\mathfrak{sl}_2$

Mentor:
Dr. Yining Zhang

Authors:
Eliot Hodges Dion Mann Evan Senkoff

13 May 2021

Table of Contents

1 Poisson Algebras

Throughout, k is a field of characteristic 0. A commutative algebra over k is denoted ComAlg_k and an arbitrary associative one by Alg_k.

1.1 Commutative Poisson Algebras

Let $A \in \mathrm{ComAlg}_k$ have a Lie bracket, a bilinear operation $[-,-] : A \times A \to A$ that satisfies

1. Skew-symmetry: $[a,b] = -[b,a]$.
2. Jacobi Identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

If, in addition, the Lie bracket on A satisfies the *Leibniz property* $[a,bc] = [a,b]c + b[a,c]$, then A is called a **Poisson algebra.**

Once we define Lie brackets on generators of an algebra A, if A is Poisson, we know how they work for the whole algebra, since we may repeatedly apply the bilinearity of the Lie bracket and the Leibniz property.

Ex. Defining a Lie bracket on $\mathbb{R}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Since we can build any polynomial as linear combinations of products of generators, we just need to know the generator brackets (since the Lie bracket is bilinear). Define:

$$[x_i,y_j] = \delta_{ij} \qquad [x_i,x_j] = 0 \qquad [y_i,y_j] = 0$$

and extend so that the bracket is bilinear, antisymmetric, and satisfies the Leibniz property. Now consider generators a,b,c, and note

$$[a,1] = [a,1\cdot 1] = [a,1]\cdot 1 + 1\cdot [a,1] = 2[a,1].$$

Since k has characteristic 0, $[a,1] = 0$. To verify this is indeed a Poisson bracket, we just need to show the Jacobi identity. Assume that

$$[a_1,[b,c]] + [b,[c,a_1]] + [c,[a_1,b]] = 0$$

$$[a_2,[b,c]] + [b,[c,a_2]] + [c,[a_2,b]] = 0$$

It suffices to show that

$$[a_1a_2,[b,c]] + [b,[c,a_1a_2]] + [c,[a_1a_2,b]] = 0,$$

since any $p \in \mathbb{R}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ can be built using generators (can do a formal proof using induction, if preferred). The computation is long, but may be done by repeated expansion using Leibniz and after noting that $[a_1a_2,b] = a_1[a_2,b] + [a_1,b]a_2$ by skew-symmetry.

Ex. The above Poisson bracket has a nice generalization which may be applied to all functions on $\mathbb{R}^{2n}$, not just polynomials. It is as follows: Let $f,g : \mathbb{R}^{2n} \to \mathbb{R}$ be real-valued functions, and suppose the coordinates of $\mathbb{R}^{2n}$ are $(x_1,\ldots,x_n,y_1,\ldots,y_n)$. Then,

$$[f,g] := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.$$

Plugging in x_i,y_j; x_i,x_j; y_i,y_j for f and g respectively shows that this bracket is indeed the same as the one we defined on $\mathbb{R}[x_1,\ldots,x_n,y_1,\ldots,y_n]$, except that it is more general. This bracket is often called the "canonical" Poisson bracket.

1.2 Generalization to Arbitrary Associative Algebras

The Poisson structure defined above is notably restricted to commutative algebras; this begs the question: can the Poisson structure defined above be extended to arbitrary associative algebras? There are multiple approaches. The "more natural" approach involves treating commutative algebras as a subset of associative algebras (which they are), and defining a Poisson structure on associative algebras that recovers the standard Poisson brackets when applied to commutative algebras. Our approach involves treating associative algebras and commutative algebras separately, i.e. viewing them independently. We shall bridge this figurative gap using representation theory.

Recall that given an associative algebra A, a **matrix representation** is an algebra homomorphism $\phi : A \to M_n(k)$. We want to consider the vector space of representations of A, $\mathrm{Rep}_n(A) = \mathrm{Hom}(A, M_n(k))$; from here, we will consider the function space on $\mathrm{Rep}_n(A)$. To better illustrate how this can be used to bridge our figurative gap, we work an informal example:

Ex. Let $A = k[t]$, and let $\phi : A \to M_n(k)$ be an algebra homomorphism, so that $\phi \in \mathrm{Rep}_n(A)$. It is not difficult to see that because ϕ preserves the structure of A, ϕ is completely determined by $M := \phi(t)$, since t generates A. Thus, we can form a bijection between the set of n-dimensional representations of A and the set of $n \times n$ matrices with entries in k:

$$\mathrm{Rep}_n(A) \longleftrightarrow M_n(k) \longleftrightarrow k^{n^2}.$$

In this case, the corresponding commutative algebra would be $k[t_{ij}]_{1 \le i,j \le n}$.

More formally, for each representation ρ of A and for each $a \in A$, we get a matrix $\rho(a) \in M_n(k)$. We want to consider the functions on the representation space of A, i.e. the dual space of $\mathrm{Rep}_n(A)$ (the vector space of all linear functionals on $\mathrm{Rep}_n(A)$). Hence, it is helpful to know the dual basis for $\mathrm{Rep}_n(A)$. We return to the above setting to make things more clear:

Ex. Recall $A = k[t]$ and that the n-dimensional representation space of A can be put in bijection with $M_n(K)$. Moreover, take it on faith that the corresponding commutative Poisson algebra will be $A_n := k[t_{ij}]_{1 \le i,j \le n}$, i.e. that the set of functions on the n-dimensional representation space is $k[t_{ij}]_{1 \le i,j \le n}$, where t_{ij} corresponds to the function selecting the ij-coordinate from some matrix M. Now, recall that dual basis for $M_n(k)$ is the set of functions E_{ij}^*, $1 \le i, j \le n$, such that E_{ij}^* selects the ij-coordinate of any matrix M. We shall see that each E_{ij}^* corresponds to left-multiplication by the matrix E_{ji}. Given some matrix $M := m_{11}E_{11} + \ldots + m_{nn}E_{nn} \in M_n(k)$, note that

$$\mathrm{tr}(E_{ji}M) = \mathrm{tr}(m_{i1}E_{j1} + \ldots + m_{ij}E_{jj} + \ldots + m_{in}E_{jn}) = m_{ij}.$$

Thus, because we have the correspondence (informally) shown above, we may define a Poisson bracket on A_n in the following way: Let $[t_{ij}, t_{kl}] = \delta_{il}t_{kj} - \delta_{kj}t_{il}$. It can (and should) be verified that this is indeed a Poisson bracket on A_n—it is indeed skew-symmetric: $[t_{kl}, t_{ij}] = \delta_{kj}t_{il} - \delta_{il}t_{kj} = -[t_{ij}, t_{kl}]$. As to why we should define the bracket in this way has not yet been discussed; the reason is as follows: As we noted above, t_{ij} corresponds to E_{ji} in $M_n(k)$. Thus, it is natural to use this correspondence to define our bracket. Indeed, because there is a standard bracket defined for the general Lie algebra $\mathfrak{gl}_n(k)$ ($[a, b] = ab - ba$ for $a, b \in M_n(k)$),

$$[t_{ij}, t_{kl}] \longleftrightarrow [E_{ji}, E_{lk}] = E_{ji}E_{lk} - E_{lk}E_{ji} = \delta_{il}E_{jk} - \delta_{kj}E_{li} \longleftrightarrow \delta_{il}t_{kj} - \delta_{kj}t_{il}.$$

Thus, it is not difficult to see that A_n, the functions on the representation space, must indeed be $k[t_{ij}]_{1 \le i,j \le n}$, on which we already know how to define a Poisson bracket.

Generally, we are not so lucky, and A tends to have more than one generator. In this case, $A_n = k[\mathrm{Rep}_n(A)]$ (the coordinate ring of the representation space, with multiplication given by the usual multiplication of maps from $M_n(k) \to k$) is generated by the usual a_{ij} where $a \in A$ and $1 \le i, j \le n$ and is constrained by the following:

$$(ab)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad (\lambda a + \mu b)_{ij} = \lambda a_{ij} + \mu b_{ij}$$

for all $a, b \in A$ and $\lambda, \mu \in k$. Note that these constraints are the familiar ones given by matrix multiplication. We illustrate this with another simple example, where A is the free algebra on two letters.

Ex. Let $A = k\langle x, y\rangle$, the free algebra on the set $\{x, y\}$. Informally, one can think of A similarly to $k[x, y]$, but noncommutative (note that $k\langle t\rangle \cong k[t]$). Now fix an $a \in A$ and define a matrix-valued function on $\mathrm{Rep}_n(A)$ called $a_{ij} : \rho \mapsto \rho(a)_{ij}$; in other words, a_{ij} takes the ij-th entry of the matrix $\rho(a) \in M_n(k)$ for $\rho \in \mathrm{Rep}_n(A)$. If we have the matrix representations

$$x \mapsto \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad \text{and} \qquad y \mapsto \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

then

$$xy \mapsto \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

and, in the general nth case, the function $(xy)_{ij}$ is given by the linear algebra formula

$$(xy)_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}.$$

The corresponding commutative algebra is $A_n := k[x_{ij}, y_{ij}]$.

Ex. Ironically, when $A = k[x, y]$ (note A is commutative), A_n is not a polynomial algebra, but instead a quotient of one. This is due to various constraints induced by the commutativity of A. Suppose $X = (x_{ij}), Y = (y_{ij}) \in M_n(k)$ are n-dimensional representations of x and y. Thus, $A_n = k[\mathrm{Rep}_n(A)]$ is generated by x_{ij} and y_{ij} subject to the relation

$$(xy)_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj} = \sum_{k=1}^{n} y_{ik} x_{kj} = (yx)_{ij}.$$

Therefore, the corresponding commutative algebra becomes

$$A_n := k[x_{ij}, y_{ij}] / \Big(\big(\textstyle\sum_{k=1}^{n} x_{ik} y_{kj} - \sum_{k=1}^{n} y_{ik} x_{kj} \big) \Big),$$

where $1 \leq i, j \leq n$.

1.3 H_0-Poisson Algebras

The definition of a Poisson structure for a commutative algebra (Lie bracket satisfying the Leibniz rule) can be applied to any noncommutative algebra as well. However, as noted in Crawley-Boevey, the only such Poisson brackets on "a genuinely noncommutative prime ring" are the commutator bracket $[a, b] = ab - ba$ and its multiples. In this section we define the notion of an *"H_0-Poisson structure"*, which is the weakest known structure that induces a Poisson structure on A_n for any associative algebra A.

First, we begin with a few definitions.

Definition 1.1. Let L be a Lie algebra. A derivation $d : L \to L$ is a linear map satisfying the Leibniz law:

$$d([a, b]) = [a, d(b)] + [d(a), b]$$

for all $a, b \in L$. In the context of associative algebras, if $d : A \to A$ is in $\mathrm{Der}(A)$, then for all $a, b \in A$,

$$d(ab) = d(a)b + ad(b).$$

Definition 1.2. Let A be an associative algebra. Define $A_\natural = A/[A, A]$, where $[A, A]$ is the span of all commutators, i.e. $[A, A] = \mathrm{span}\{ab - ba \mid a, b \in A\}$. Define $A_{\natural\natural} = A/([A, A])$ to be quotient of A by the two-sided ideal $([A, A])$.

Note that $[A,A]$ is not necessarily a two-sided ideal of A, and therefore $A_\natural$ is *not* necessarily an associative algebra itself, since it will not in general have a well-defined multiplication. Herein lies the distinction between $A_\natural$ and $A_{\natural\natural}$: $A_\natural$ is the quotient of A by the subspace $[A,A]$, and while it is a vector space, it is not an algebra. On the other hand, $A_{\natural\natural}$ is a commutative algebra—in a sense, modding out by $([A,A])$ "abelianizes" A (note that $A_{\natural\natural}$ is indeed an algebra, since $([A,A])$ is a two-sided ideal of A; it is not difficult to see that it is commutative). But, since the ideal in question ends up being so large, $A_{\natural\natural}$ ends up not being very interesting.

Now, for each $a \in A$ let $\bar{a} = \pi(a)$, where $\pi : A \to A/[A,A]$ is the canonical projection map. Note that if $d \in \mathrm{Der}(A)$, d induces a linear map $\bar{d} : A_\natural \to A_\natural$ defined by $\bar{a} \mapsto \overline{d(a)}$: because

$$d([x,y]) = [x, d(y)] + [d(x), y] \in [A,A]$$

for all $x, y \in A$ it follows that if $\bar{a} = \bar{b} \in A_\natural$ so that $a - b \in [A,A]$, then

$$\bar{d}(\bar{a}) - \bar{d}(\bar{b}) = \overline{d(a)} - \overline{d(b)} = \pi(d(a) - d(b)) = \pi(d(a-b)) = \overline{d(a-b)} = 0$$

where the last equality follows from the fact that $a-b \in [A,A]$ so that $a-b$ is a linear combination of commutators and therefore $d(a-b) \in [A,A]$ by the linearity of d. Thus, the induced map $\bar{d}$ is well-defined, its linearity being inherited from d.

Definition 1.3. An H_0-Poisson structure on A, an arbitrary associative algebra, is a Lie bracket $\langle -, - \rangle$ on $A_\natural$ such that for all $a \in A$, the linear map $\langle \bar{a}, - \rangle : A_\natural \to A_\natural$ is induced by some derivation $d_a : A \to A$. To be more precise: it is $\langle \bar{a}, - \rangle$ is the induced linear map $\bar{d_a}$ we discussed above.

Note. If A is commutative, then $[A,A] = 0$ so that $A_\natural = A/0 = A$. In this case, it is not difficult to see that an H_0-Poisson structure on A is precisely a Poisson bracket on A; we only need to verify the Leibniz rule. But, doing so is easy:

$$\langle a, bc \rangle = d_a(bc) = d_a(b)c + bd_a(c) = \langle a, b \rangle c + b\langle a, c \rangle$$

for all $a, b, c \in A$ (we have dropped the $^{-}$ notation here, as it is unnecessary).

Note. For a group G and (finite-dim) vector space V, a representation of G in V is a homomorphism $\phi : G \to \mathrm{GL}(V)$. This ϕ induces actions of G on V. We can express actions of G on $k[V]$ by the relationship

$$(g \cdot f)(x) = f(g^{-1} \cdot x) = f(x \cdot g)$$

where $f \in k[V]$ and $x \in V$. In other words, left-action of g on $f \in k[V]$ gives right-action of g on $x \in V$. The question is: what subspace of $k[V]$ is G-invariant, i.e., $g \cdot f = f$ for all $g \in G$? This subspace is denoted $k[V]^G$.

Sometimes, it is more fruitful to consider the equivalence classes of representations of A, which we denote by $\mathrm{Rep}_n(A)/GL_n$. Recall that representations on conjugacy classes of $M_n(k)$ are equivalent, since two representations $\phi_1 : A \to \mathrm{End}(V_1)$ and $\phi_2 : A \to \mathrm{End}(V_2)$ are equivalent if there is a vector space isomorphism $\psi : V_1 \to V_2$ such that $\phi_1(a) = \psi^{-1}\phi_2(a)\psi$ for all $a \in A$ (this induces a module isomorphism between corresponding A-modules V_1 and V_2). Translating the above into the language of matrices and matrix representations should not be difficult. From here on out, we let $A_n^{GL_n} = k[\mathrm{Rep}_n(A)]^{GL_n}$, the coordinate ring of invariants (i.e. of equivalence classes of representations). Because equivalence classes of representations are invariant under conjugation by any element $g \in GL_n$, it is natural here to define the GL_n-action to be

$$g \cdot m_{ij}(M) = m_{ij}(g^{-1}Mg) = \sum_{k,l=1}^{n} (g^{-1})_{ik} m_{kl} g_{lj}$$

for some $m \in M_n(k)$. We show that $\mathrm{tr}(m) := \sum_{i=1}^{n} m_{ii} \in A_n$ is one such invariant under the action of GL_n; in fact the elements $\mathrm{tr}(m)$ generate $A_n^{GL_n}$ (see reference in the Crawley-Boevey paper). We see that

$$g \cdot \mathrm{tr}(m) = \sum_{i=1}^{n} g \cdot m_{ii} = \sum_{i=1}^{n} \sum_{k,l=1}^{n} (g^{-1})_{ik} m_{kl} g_{li} = \sum_{i,k,l=1}^{n} m_{kl} g_{li} (g^{-1})_{ik} = \sum_{k,l=1}^{n} \delta_{kl} m_{kl} = \mathrm{tr}(m),$$

so $\mathrm{tr}(m)$ is indeed invariant for all m. Now, the motivation for considering the equivalence classes of representations of A is best illustrated by the following theorem due to Crawley-Boevey:

Theorem 1.4. *If A is a finitely generated associative algebra over an algebraically closed field k of characteristic 0, $\langle -,- \rangle$ is an H_0-Poisson structure and n a natural number, then there is a unique Poisson structure $\{-,-\}$ on $A_n^{GL_n}$ with the property*

$$\{\mathrm{tr}(a), \mathrm{tr}(b)\} = \mathrm{tr}\langle \bar{a}, \bar{b} \rangle$$

for all $a, b \in A$.

Note that in the above, $\bar{a}, \bar{b}$ are as discussed in 1.3. Proof of this theorem is given in the aforementioned paper by Crawley-Boevey. Thus, we see that an H_0-Poisson structure on A induces a unique Poisson structure on $A_n^{GL_n}$ in the classical sense:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_n^{GL_n} \\ A_\natural \arrow[ur] & \end{tikzcd}$$

2 Double Poisson Brackets

2.1 Tensor Products and Algebras

Let V, W be vector spaces over k.

Definition 2.1. The **tensor product** $V \otimes_k W$ is a vector space over k generated by $v \otimes w$ satisfying

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$.
3. $(kv) \otimes w = k(v \otimes w) = v \otimes (kw)$.

Note. Writing $a \in V \otimes W$ or $a = a' \otimes a''$ is understood as a linear combination of vectors of the form $v \otimes w$.

Definition 2.2. An ***n*-fold tensor product** is the vector space

$$V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n\text{-copies}}.$$

Note. Write the vector $v_1 \otimes \cdots \otimes v_n$ as $(v_1, \ldots, v_n)$.

If $\dim V = n$ and $\dim W = m$, then $\dim(V \otimes W) = nm$. Intuitively, this is because $v_i \otimes w_j$ form a basis for $V \otimes W$, given that v_i and w_j are basis vectors for V and W.

Definition 2.3. The infinite-dimensional vector space

$$T(V) = k \otimes V \otimes V^{\otimes 2} \otimes \cdots \otimes V^{\otimes n} \otimes \cdots$$

can be made into an algebra using the following product, often referred to as the concatenation product: if $u := (v_1, \ldots, v_p) \in V^{\otimes^p}$ and $v := (v_{p+1}, \ldots, v_n) \in V^{\otimes^{n-p}}$, then $uv := (v_1, \ldots, v_n) \in V^{\otimes^n}$. Extending bilinearly, we get a well-defined multiplication on $T(V)$ so that $T(V)$ becomes an algebra. We call this the **tensor algebra generated by** V.

The tensor algebra generated by V, $T(V)$, is also known as the free algebra on the vector space V; in fact, it's the free algebra generated by basis vectors of V. For example, we can view the algebra $k\langle x, y\rangle$ as a tensor algebra $T(V)$, where V is generated by x and y. In this way, $T(V)$ is the noncommutative version of a polynomial algebra.

In general, $a \otimes b \neq b \otimes a$, but we can define a map $(-)^\circ : A \otimes A \to A \otimes A$ where $(a \otimes b)^\circ = b \otimes a$. Alternatively, we see that the symmetric group S_2 acts on $A \otimes A$, where the group action is given by $(12) \cdot (a \otimes b) := (a \otimes b)^\circ$ and extending linearly (this is easy enough to verify). Even more generally, we see that given the n-fold tensor product $A^{\otimes^n}$, $S_n \curvearrowright A^{\otimes^n}$ in the natural way—the generalization to the n-dimensional case is easy to see. For example, $S^3 \curvearrowright A^{\otimes^3}$, and $(123)(a \otimes b \otimes c) = c \otimes a \otimes b$.

Recall the definition of a left module and a right module, and recall that a bimodule is a module that is both a left and a right module. Now, note that in addition to the S_n-action on $A^{\otimes^n}$, there is also an A-action called the outer bimodule structure. It is given by

$$x(a_1 \otimes \ldots \otimes a_n)y = xa_1 \otimes \ldots \otimes a_n y \quad \text{for all } x, y \in A \text{ and } a_1 \otimes \ldots \otimes a_n \in A^{\otimes^n}.$$

For $n = 2$, there is an additional A-action on $A^{\otimes^2}$ given by $x * (a_1 \otimes a_2) * y := a_1 y \otimes x a_2$. This action is often referred to as the *inner action*.

2.2 n-brackets

As discussed previously, our goal is to construct a Poisson structure on $A_n := k[\mathrm{Rep}_n(A)]$, the coordinate algebra of the representation space of A. The introduction of H_0-Poisson algebras makes this possible, but only up to GL_n-invariance. In other words, recall that an H_0-Poisson structure on A induces a Poisson structure on $A_n^{GL_n}$, but not A_n itself. Note that if we tried to define a Poisson bracket on A_n we would immediately run into difficulty. We would want to define our structure on generators of A_n, which, recall, are the functions $a_{ij} \in k[\mathrm{Rep}_n(A)]$ for all $a \in A$; we would need to define $[a_{ij}, b_{kl}]$ for $a, b \in A$. However, note that this would depend on four indices, instead of two as desired. As a remedy to this problem, we turn to tensor products and develop the notion of a double Poisson bracket.

Definition 2.4. Suppose A is an associative k-algebra and $\tau_{(-)} \in S_n$ are permutations. An ***n*-bracket** is a linear map $\{\!\{-, \dots, -\}\!\} : A^{\otimes^n} \to A^{\otimes^n}$ satisfying

1. $\{\!\{a_1, \dots, a_n a_n'\}\!\} = \{\!\{a_1, \dots, a_n\}\!\} a_n' + a_n \{\!\{a_1, \dots, a_n'\}\!\}$
2. $\tau_{(1\dots n)} \circ \{\!\{-, \dots, -\}\!\} \circ \tau_{(1\dots n)}^{-1} = (-1)^{n+1} \{\!\{-, \dots, -\}\!\}$

Example 2.5. Setting $n = 1$, we see that a 1-bracket is just a derivation. When $n = 2$, we get a 2-bracket, which we shall refer to as a "double bracket." Note that in this case, 1. and 2. above may be interpreted as

$$\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\}c + b\{\!\{a, c\}\!\} = \{\!\{a, b\}\!\}' \otimes \{\!\{a, b\}\!\}''c + b\{\!\{a, c\}\!\}' \otimes \{\!\{a, c\}\!\}''$$

and $\{\!\{b, a\}\!\} = -\{\!\{a, b\}\!\}^\circ$. Moreover, it is simple to verify that these properties imply

$$\{\!\{ab, c\}\!\} = -\{\!\{c, ab\}\!\}^\circ = -\{\!\{c, a\}\!\}''b \otimes \{\!\{c, a\}\!\}' - \{\!\{c, b\}\!\}'' \otimes a\{\!\{c, b\}\!\}' = \{\!\{a, c\}\!\} * b + a * \{\!\{b, c\}\!\},$$

where the last equality follows from applying the inner bimodule structure given above and from the fact that $\{\!\{a, c\}\!\} \in A \otimes A$ may be written as $\{\!\{a, c\}\!\}' \otimes \{\!\{a, c\}\!\}'' = -\{\!\{c, a\}\!\}'' \otimes \{\!\{c, a\}\!\}'$.

2.3 Coalgebras: A brief, informal Introduction

Recall that in an algebra A, there is a multiplication $\cdot : A \times A \to A$. Coalgebras extend the sort of natural converse of this idea—they are equipped with a comultiplication $\Delta : A^* \to A^* \otimes A^*$, where A^* denotes the dual space of A. There is a relatively natural way to define comultiplication in a coalgebra, but as we are treating this section informally, we opt to give a familiar example instead of the formal definition.

Example 2.6. Suppose $A = M_n(k)$. We want to define $\Delta : M_n(k)^* \to M_n(k)^* \otimes M_n(k)^* = (M_n(k) \otimes M_n(k))^*$ (the equality only holds with finite-dimensional algebras). If $f \in M_n(k)^*$, then $\Delta(f)$ will be a function $M_n(k) \otimes M_n(k) \to k$. Thus, we see that we need to determine how $\Delta(f)$ acts on elements of $M_n(k) \otimes M_n(k)$. If $X \otimes Y \in M_n(k) \otimes M_n(k)$, then it is natural to let $\Delta(f)(X \otimes Y) = f(XY)$. We see how this works for the generators of $M_n(k)^*$, which, recall, are the matrices E_{ij}^*, $1 \le i, j \le n$, selecting the ij-entry of a matrix. In this case, we may write

$$\Delta(E_{ij}^*) = \sum a_{k_1 l_1 k_2 l_2} E_{k_1 l_1}^* \otimes E_{k_2 l_2}^*,$$

where the k's and l's range from 1 to n. Moreover,

$$(XY)_{ij} = \sum_k X_{ik} Y_{kj} = \sum a_{k_1 l_1 k_2 l_2} X_{k_1 l_1} Y_{k_2 l_2}$$

only if $k_1 = i$, $l_2 = j$, and $k_2 = l_1$ and if

$$a_{k_1 l_1 k_2 l_2} = \begin{cases} 1 & \text{if } k_1 = i,\ l_2 = j, \text{ and } k_2 = l_1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, it is not difficult to see that

$$\Delta(E_{ij}^*) = \sum_k E_{ik}^* \otimes E_{kj}^* = \Delta(E_{ij}^*)' \otimes \Delta(E_{ij}^*)''.$$

Thus we see that any finite dimensional algebra gives a coalgebra via its linear dual. Moreover, the above example gives us good practice working with tensors, specifically the $'$ and $''$ notation.

Note. It is pretty easily shown that $(V \otimes W)^* \cong V^* \otimes W^*$ when both vector spaces are finite dimensional. The proof is as follows:

Proof. We show that there exists an injection $V^* \otimes W^* \to (V \otimes W)^*$; we can use dimensions to force the equality. First note that $\dim(V \otimes W)^* = \dim V \cdot \dim W = \dim(V^* \otimes W^*)$. Then, let $\phi : V^* \otimes W^* \to (V \otimes W)^*$ be given by $\phi(f \otimes g) = f \cdot g$, where $f \cdot g$ is the element of $(V \otimes W)^*$ such that $f \cdot g(v \otimes w) = f(v)g(w)$ (the motivation for this is because $(f \otimes g)(v \otimes w) = f(v)g(w)$). Now, $f \otimes g$ is nonzero if neither f nor g is 0, so $f \cdot g$ cannot be 0 either if $f \otimes g \neq 0$. Thus ϕ is an injection, and since the vector spaces have the same dimension, it follows that $(V \otimes W)^* \cong V^* \otimes W^*$. $\square$

2.4 Properties of Double Brackets

We will show that any double bracket on an algebra A is completely determined by how it operates on the generators of A. The linearity of n-brackets ensures that we only need to check that double brackets of products of generators can be written in terms of double brackets of generators. We begin with the case $\{\!\{x_1x_2, y_1y_2\}\!\}$ and then generalize to the case with n-products in each slot. We do this by repeatedly applying the properties listed in 2.4 and 2.5. We see that

$$\begin{aligned}
\{\!\{x_1x_2, y_1y_2\}\!\} &= y_1\{\!\{x_1x_2, y_2\}\!\} + \{\!\{x_1x_2, y_1\}\!\}y_2 \\
&= y_1(\{\!\{x_1, y_2\}\!\} * x_2 + x_1 * \{\!\{x_2, y_2\}\!\}) \\
&\qquad + (\{\!\{x_1, y_1\}\!\} * x_2 + x_1 * \{\!\{x_2, y_1\}\!\})y_2 \\
&= y_1\{\!\{x_1, y_2\}\!\}'x_2 \otimes \{\!\{x_1, y_2\}\!\}'' + y_1\{\!\{x_2, y_2\}\!\}' \otimes x_1\{\!\{x_2, y_2\}\!\}'' \\
&\qquad + \{\!\{x_1, y_1\}\!\}'x_2 \otimes \{\!\{x_1, y_1\}\!\}''y_2 + \{\!\{x_2, y_1\}\!\}' \otimes x_1\{\!\{x_2, y_1\}\!\}''y_2
\end{aligned}$$

as desired. The n-fold case is similar:

$$\begin{aligned}
&\{\!\{x_1 \cdots x_n, y_1 \cdots y_n\}\!\} \\
&\quad = \sum_{1 \le i,j \le n} y_1 \cdots y_{j-1}\{\!\{x_i, y_j\}\!\}'x_{i+1} \cdots x_n \otimes x_1 \cdots x_{i-1}\{\!\{x_i, y_j\}\!\}''y_{j+1} \cdots y_n \\
&\quad = -\sum_{1 \le i,j \le n} y_1 \cdots y_{j-1}\{\!\{y_j, x_i\}\!\}''x_{i+1} \cdots x_n \otimes x_1 \cdots x_{i-1}\{\!\{y_j, x_i\}\!\}'y_{j+1} \cdots y_n \\
&\quad = -\left(\sum_{1 \le i,j \le n} x_1 \cdots x_{i-1}\{\!\{y_j, x_i\}\!\}'y_{j+1} \cdots y_n \otimes y_1 \cdots y_{j-1}\{\!\{y_j, x_i\}\!\}''x_{i+1} \cdots x_n\right)^\circ \\
&\quad = -\{\!\{y_1 \cdots y_n, x_1 \cdots x_n\}\!\}^\circ
\end{aligned}$$

2.5 Double Poisson Brackets

Recall the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

where $[-, -] : A \otimes A \to A$. Note that the Jacobi identity may be written as

$$[-, [-, -]] + [-, [-, -]] \circ \tau_{(123)}^{-1} + [-, [-, -]] \circ \tau_{(132)}^{-1} = 0,$$

and that $[-, [-, -]]$ is a function first from $A \otimes A \otimes A \to A \otimes A$ and then from $A \otimes A \to A$. To extend the notion of a Poisson bracket to higher-dimensional tensors, we need to develop the notion of a higher-dimensional Jacobi identity. Specifically, we already have developed the notion of a double Poisson bracket, and it would follow that a "Double Poisson Algebra" would be an algebra on which a double bracket is defined such that the double bracket satisfies a "Double Jacobi Identity." We shall see that this is indeed the case. However, note that we run into a bit of trouble if we define such a double Jacobi identity naively. Indeed, we see that $\{\!\{-, \{\!\{-, -\}\!\}\}\!\}$ doesn't even make sense, since the element in the second slot would be a 2-tensor and since $\{\!\{-, -\}\!\} : A \otimes A \to A \otimes A$. To remedy this problem, we make the following definition:

Definition 2.7. Let $a \in A$ and $b_1 \otimes b_2 \ldots \otimes b_n \in A^{\otimes^n}$ for an associative k-algebra A. Then, we define the following:

$$\{\!\{a, b_1 \otimes \ldots \otimes b_n\}\!\}_L := \{\!\{a_1, b_1\}\!\} \otimes b_2 \otimes \ldots \otimes b_n$$

and

$$\{\!\{a, b_1 \otimes \ldots \otimes b_n\}\!\}_R := b_1 \otimes \ldots \otimes b_{n-1} \otimes \{\!\{a, b_n\}\!\}.$$

Note that $\{\!\{a, b_1 \otimes \ldots \otimes b_n\}\!\}_L$ and $\{\!\{a, b_1 \otimes \ldots \otimes b_n\}\!\}_R$ are both elements of $A^{\otimes^{n+1}}$. Thus, we see that $\{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L$ and $\{\!\{a, \{\!\{b, c\}\!\}\}\!\}_R$ are both elements of $A \otimes A \otimes A$. We are now fully equipped to define the so-called Double Jacobi Identity.

Definition 2.8. The *Double Jacobi Identity* is as follows: Given a double bracket $\{\!\{-, -\}\!\}$ on some algebra A, $\{\!\{-, -\}\!\}$ satisfies the Double Jacobi Identity if

$$\{\!\{-, \{\!\{-, -\}\!\}\}\!\}_L + \tau_{(123)} \circ \{\!\{, \{\!\{-, -\}\!\}\}\!\}_L \circ \tau_{(123)}^{-1} + \tau_{(132)} \circ \{\!\{-, \{\!\{-, -\}\!\}\}\!\}_L \circ \tau_{(132)}^{-1} = 0.$$

We let

$$\{\!\{a, b, c\}\!\} = \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + \tau_{(123)} \circ \{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L + \tau_{(132)} \circ \{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L$$

for some $a, b, c \in A$

Proposition 2.9. *The above bracket $\{\!\{a, b, c\}\!\}$ is a triple bracket.*

See Proposition 2.3.1 in van den Bergh's paper on Double Poisson algebras for the proof. Note though that this is indeed a bracket from $A^{\otimes^3} \to A^{\otimes^3}$.

Definition 2.10. A Double Poisson algebra is an algebra A equipped with a double bracket satisfying the double Jacobi identity, i.e. $\{\!\{-, -, -\}\!\} = 0$.

It suffices to show that $\{\!\{-, -, -\}\!\} = 0$ only on generators of A. This follows because $\{\!\{ab, c, d\}\!\} = \tau_{(123)}\{\!\{c, d, ab\}\!\}$ and because $\{\!\{a, bc, d\}\!\} = \tau_{(132)}\{\!\{d, a, bc\}\!\}$. Moreover, we have $\{\!\{x, 1\}\!\} = \{\!\{x, 1\}\!\} + \{\!\{x, 1\}\!\} = 2\{\!\{x, 1\}\!\}$, implying $\{\!\{x, 1\}\!\} = 0$, so that $\{\!\{1, x\}\!\} = -\{\!\{x, 1\}\!\}^\circ = 0$. Finally, we return again to $A = k[t]$ and construct some double brackets on this algebra.

Example 2.11. As we showed above, to construct a double bracket on A, we only need to define it on t. From there, we may use the higher dimensional Leibniz rule to extend the bracket. Thus, all we will have to check is that $\{\!\{t, t\}\!\} = -\{\!\{t, t\}\!\}^\circ$. One such double bracket on A is given by $\{\!\{t, t\}\!\} = t \otimes 1 - 1 \otimes t$. We will show that this is indeed a double bracket; that it further induces a double Poisson structure on A. Verifying that this is indeed a double bracket is simple:

$$\{\!\{t, t\}\!\} = t \otimes 1 - 1 \otimes t = -(1 \otimes t - t \otimes 1) = -(t \otimes 1 - 1 \otimes t)^\circ = -\{\!\{t, t\}\!\}^\circ.$$

By definition, $\{\!\{-, -\}\!\}$ satisfies the Leibniz rule for double brackets, so both properties hold and it is indeed a double bracket. All that remains to prove is that A with this structure is a double Poisson algebra. We just have to check that the double Jacobi identity is satisfied, i.e. that $\{\!\{t, t, t\}\!\} = 0$. To do this, we first compute $\{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L$, since the other terms are just permutations applied to this expression. Hence, we see that

$$\{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L = \{\!\{t, t \otimes 1 - 1 \otimes t\}\!\} = \{\!\{t, t\}\!\} \otimes 1 - \{\!\{t, 1\}\!\} \otimes t = (t \otimes 1 - 1 \otimes t) \otimes 1 = t \otimes 1 \otimes 1 - 1 \otimes t \otimes 1,$$

so

$$\tau_{(123)} \circ \{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L = 1 \otimes t \otimes 1 - 1 \otimes 1 \otimes t$$

and

$$\tau_{(132)} \circ \{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L = 1 \otimes 1 \otimes t - t \otimes 1 \otimes 1.$$

The sum of these three expressions is 0, so we have $\{\!\{t, t, t\}\!\} = 0$ as desired. Therefore, $A = k[t]$ equipped with the double Poisson bracket given above makes A into a double Poisson algebra. Another such bracket with this property is $\{\!\{t, t\}\!\} = t^2 \otimes t - t \otimes t^2$.

3 Double Poisson Bracket and the H_0-Poisson Bracket

3.1 Poisson Homomorphisms

Definition 3.1. Suppose A and B are Poisson algebras with brackets $\{-,-\}$ and $[-,-]$ respectively. A Poisson homomorphism is a map $\phi : A \to B$ such that for all $x, y \in A$

$$\phi(\{x,y\}) = [\phi(x), \phi(y)].$$

Similarly, if A, B are double Poisson algebras with brackets $\{\!\{-,-\}\!\}_A$ and $\{\!\{-.-\}\!\}_B$, then a double Poisson homomorphism is a map $\phi : A \to B$ such that

$$\phi(\{\!\{x,y\}\!\}_A) = \{\!\{\phi(x), \phi(y)\}\!\}_B$$

for all $x \otimes y \in A \otimes A$.

3.2 The Noncommutative, Double-Bracket Analog of the Classical Poisson Structure on $\mathbb{R}^{2n}$

Recall that we may define a Poisson structure on the algebra of smooth functions mapping R^{2n} to itself: $\{f,g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$, where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are the coordinates in $\mathbb{R}^{2n}$. To define an analogous noncommutative double bracket structure, we consider the tensor algebra over a $2n$-dimensional vector space V. Recall that $T(V) \cong k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$. Here, we let $\{\!\{x_i, x_j\}\!\} = 0 = \{\!\{y_i, y_j\}\!\}$, and $\{\!\{x_i, y_j\}\!\} = \delta_{ij} \cdot 1 \otimes 1 = -\{\!\{y_j, x_i\}\!\}$. Note that $\{\!\{-,-,-\}\!\} = 0$ is then true automatically, making $T(V)$ a double Poisson algebra.

3.3 An Alternate Definition of an H_0-Poisson Algebra

We compile here a list of previously defined terms for convenient reference.

Definition 3.2. For an associative k-algebra A, define $A_\natural$ by $A_\natural := A/[A,A]$.

Remark 3.3. Observe that $A_\natural$ is indeed a vector space, but not necessarily an algebra.

Definition 3.4. We denote by $\mathrm{Der}(A)$ the vector space of all derivations on A. That is, the space of linear maps $f\colon A \to A$ such that $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. A map $g_a\colon A \to A$ is an *inner derivation* in case $g_a(b) = ab - ba$ for some $a \in A$. The space of all inner derivations (Inn(A)) forms a subspace of Der(A).

These definitions allow us an alternate way to define an H_0 Poisson bracket on A.

Definition 3.5. An H_0-*Poisson Bracket on* A is a linear map $\rho\colon A/[A,A] \to \mathrm{Der}(A)/\mathrm{Inn}(A)$ such that $\{\bar{a}, \bar{b}\} = \overline{\rho(\bar{a})^\sim(b)}$ where $\rho(\bar{a})^\sim$ is any map from $A/[A,A]$ to $\mathrm{Der}(A)$ whose composition with the the natural projection map $\pi\colon \mathrm{Der}(A) \to \mathrm{Der}(A)/\mathrm{Inn}(A)$ is equal to ρ.

3.4 The Relationship between Double Brackets and H_0-Poisson Brackets

Suppose A is double Poisson. Then, note that we have the following chain of maps: $m \circ \{\!\{-,-\}\!\} : A \otimes A \to A \otimes A \to A$, where m is the linear multiplication map under which $x \otimes y \mapsto xy$. Now, we attempt to define a new bracket $\{-,-\} : A \otimes A \to A$, given by $\{a,b\} = m \circ \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}'\{\!\{a,b\}\!\}''$. Since

$$\{\!\{a,b,c\}\!\} = b\{\!\{a,c\}\!\}' \otimes \{\!\{a,c\}\!\}'' + \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''c,$$

we have

$$\{a,bc\} = b\{\!\{a,c\}\!\}'\{\!\{a,c\}\!\}'' + \{\!\{a,b\}\!\}'\{\!\{a,b\}\!\}''c = b\{a,c\} + \{a,b\}c, \tag{1}$$

and thus the single bracket indeed satisfies the Leibniz rule. However we see that $\{-,-\}$ is not antisymmetric:
$$\{a,b\} = -\{\!\{b,a\}\!\}''\{\!\{b,a\}\!\}' \neq -\{\!\{b,a\}\!\}'\{\!\{b,a\}\!\}'' = -\{b,a\}.$$
However, in $A_\natural$,
$$\overline{\{a,b\}} = -\overline{\{\!\{b,a\}\!\}''\{\!\{b,a\}\!\}'} = -\overline{\{\!\{b,a\}\!\}'\{\!\{b,a\}\!\}''} = -\overline{\{b,a\}}.$$
Thus, we must go one more step, and let $\{-,-\} : A \otimes A \to A_\natural$ be given by $\{-,-\} = \pi \circ m \circ \{\!\{-,-\}\!\}$, where π is the canonical projection map from A to A. To make this into a Lie bracket on $A_\natural$, we need to project each of the

Proposition 3.6. *As given above, $\{-,-\}$ induces well-defined maps $A_\natural \otimes A \to A$ and $A_\natural \otimes A_\natural \to A_\natural$, the second of which being antisymmetric.*

Proof. We have antisymmetry of the second by the above. We begin by noting $\{bc,a\} = \{cb,a\}$. Recall that
$$\{\!\{bc,a\}\!\} = b * \{\!\{c,a\}\!\} + \{\!\{b,a\}\!\} * c = \{\!\{c,a\}\!\}' \otimes b\{\!\{c,a\}\!\}'' + \{\!\{b,a\}\!\}'c \otimes \{\!\{b,a\}\!\}''.$$
Hence,
$$\{bc,a\} = \{\!\{c,a\}\!\}'b\{\!\{c,a\}\!\}'' + \{\!\{b,a\}\!\}'c\{\!\{b,a\}\!\}'' = \{cb,a\};$$
it follows that $\{bc-cb,a\} = 0$. Thus, projecting the first argument of $\{-,-\}$ is well-defined, and we have a well-defined map $\{-,-\} : A_\natural \otimes A \to A$, which of course, we may project down to A, giving us a map from $A_\otimes A \to A$. Above we noted that in $A_\natural$, $\{-,-\}$ is antisymmetric, so $\{a, bc-cb\} = 0$ in $A_\natural$ as well. Therefore, $\{-,-\} : A_\natural \otimes A_\natural \to A_\natural$ is well-defined. □

Proposition 3.7. *$\{-,-\}$ is a Lie bracket on $A_\natural$.*

Proof. We already have that the bracket in question is well-defined and antisymmetric on $A_\natural$. In 2.4.2-2.4.6, van den Bergh shows that the double Jacobi identity implies the Jacobi identity in $A_\natural$. Thus, $A_\natural$ equipped with the bracket in question is a Lie algebra. □

Moreover, we see that $\{\bar{a},-\}$ is induced by the derivation $\{\bar{a},-\} : A_\natural \otimes A \to A$ (i.e. by lifting the second argument to A, which is well-defined by 3.6, we get a derivation on A by (1)). Therefore, the double Poisson structure on A induces an H_0-Poisson structure on A; in other words, double Poisson brackets are stronger that H_0-Poisson structures.

However, one of the main drawbacks of the double Poisson structure is that there is no obvious relation between double Poisson brackets and the classical Poisson structures we studied previously.

By the above, H_0-Poisson brackets are induced by double Poisson brackets, which begs the question: is the converse true? Can a double Poisson bracket be recovered from an H_0-Poisson structure on an algebra? The following example shows that this is not the case.

Example 3.8. Consider $A := k[x,y] = A_\natural$. Because $\{\!\{x,xy\}\!\} = \{\!\{x,yx\}\!\}$, it must be that
$$\begin{aligned} x\{\!\{x,y\}\!\} + \{\!\{x,x\}\!\}y &= \{\!\{x,y\}\!\}x + y\{\!\{x,x\}\!\} \\ x\{\!\{x,y\}\!\} - \{\!\{x,y\}\!\}x &= y\{\!\{x,x\}\!\} - \{\!\{x,x\}\!\}y \\ (x \otimes 1 - 1 \otimes x)\{\!\{x,y\}\!\} &= (y \otimes 1 - 1 \otimes y)\{\!\{x,x\}\!\}. \end{aligned}$$
So, $x \otimes 1 - 1 \otimes x$ divides $\{\!\{x,x\}\!\}$ and likewise $y \otimes 1 - 1 \otimes y$ divides $\{\!\{x,y\}\!\}$. This forces $\{x,x\} = \{x,y\} = 0$. A similar argument shows $\{y,y\} = \{y,x\} = 0$ as well. Thus, for this algebra, there are no interesting Poisson brackets induced by double Poisson brackets.

Thus, in general, nontrivial double brackets are not always induced by H_0-brackets. Therefore, we see that when commutative algebras are viewed as a subset of associative algebras, H_0-Poisson structures work well, whereas double Poisson structures are more compatible when noncommutative and commutative algebras are viewed separately.

3.5 Defining Poisson Brackets on the Coordinate Ring of $\mathrm{Rep}_n(A)$

Recall that the coordinate ring of the representation space of an algebra A, $A_n = k[\mathrm{Rep}_n(A)]$, is generated by the elements a_{ij}, where $1 \le i, j \le n$ and where $a \in A$. Moreover, the a_{ij} satisfy the following relations: for all $a, b \in A$ and $\lambda, \mu \in k$,

$$\lambda a_{ij} + \mu b_{ij} = (\lambda a + \mu b)_{ij}; \quad (ab)_{ij} = \sum_k a_{ik} b_{kj}; \quad 1_{ij} = \delta_{ij}.$$

We want to define a bracket on A_n, so we must determine the bracket on generators, i.e. we need to define $[a_{ij}, b_{kl}]$. Once we have defined the bracket on generators and checked that it satisfies the requisite properties, we can extend it using the Leibniz rule to get a bracket on the entirety of A_n. So, let A be double Poisson. Then define

$$[a_{ij}, b_{kl}] = \{\!\{a, b\}\!\}'_{kj} \cdot \{\!\{a, b\}\!\}''_{il}.$$

We shall see that this gives a Poisson structure on A_n.

Proposition 3.9. *The bracket defined above is well-defined and antisymmetric.*

Proof. Antisymmetry is easy to check: for all $a_{ij}, b_{kl} \in A_n$,

$$[b_{kl}, a_{ij}] = \{\!\{b, a\}\!\}'_{il}\{\!\{kj\}\!\}'' = -\{\!\{a, b\}\!\}''_{il}\{\!\{a, b\}\!\}'_{kj} = -[a_{ij}, b_{kl}],$$

where the middle equality follows from the corresponding "antisymmetry" property for double brackets. As for well-definededness, we just need to check that the bracket preserves the relations given above. Suppose $a, b, c \in A$. Then

$$\begin{aligned}[a_{ij}, (bc)_{kl}] = \{\!\{a, bc\}\!\}'_{kj}\{\!\{a, bc\}\!\}''_{il} &= (b\{\!\{a, c\}\!\}')_{kj}\{\!\{a, c\}\!\}''_{il} + \{\!\{a, b\}\!\}'_{kj}(\{\!\{a, b\}\!\}''c)_{il} \\ &= \sum_r b_{kr}\{\!\{a, c\}\!\}'_{rj}\{\!\{a, c\}\!\}''_{il} + \{\!\{a, b\}\!\}'_{kj}\{\!\{a, b\}\!\}_{ir}c_{rl}.\end{aligned}$$

Moreover,

$$\sum_r [a_{ij}, b_{kr}c_{rl}] = \sum_r [a_{ij}, b_{kr}]c_{rl} + b_{kr}[a_{ij}, c_{rl}] = \sum_r \{\!\{a, b\}\!\}'_{kj}\{\!\{a, b\}\!\}''_{ir}c_{rl} + b_{kr}\{\!\{a, c\}\!\}'_{kj}\{\!\{a, c\}\!\}''_{il},$$

so the bracket is indeed well-defined. □

Proposition 3.10. *The bracket defined above satisfies the Jacobi identity. Notably, we have*

$$\begin{aligned}&[a_{ij}, [b_{kl}, c_{st}]] + [b_{kl}, [c_{st}, a_{ij}]] + [c_{st}, [a_{ij}, b_{kl}]] \\ &\qquad = \{\!\{a, b, c\}\!\}'_{sj}\{\!\{a, b, c\}\!\}''_{il}\{\!\{a, b, c\}\!\}'''_{kt} - \{\!\{a, c, b\}\!\}'_{kj}\{\!\{a, c, b\}\!\}''_{it}\{\!\{a, c, b\}\!\}'''_{sl}.\end{aligned}$$

Proof. This is mostly an elaborate computation:

$$\begin{aligned}[a_{ij}, [b_{kl}, c_{st}]] &= [a_{ij}, \{\!\{b, c\}\!\}'_{sl}\{\!\{b, c\}\!\}''_{kt}] \\ &= [a_{ij}, \{\!\{b, c\}\!\}'_{sl}]\{\!\{b, c\}\!\}''_{kt} + \{\!\{b, c\}\!\}'_{sl}[a_{ij}, \{\!\{b, c\}\!\}''_{kt}] \\ &= \{\!\{a, \{\!\{b, c\}\!\}'\}\!\}'_{sj}\{\!\{a, \{\!\{b, c\}\!\}'\}\!\}''_{il}\{\!\{b, c\}\!\}''_{kt} + \{\!\{b, c\}\!\}'_{sl}\{\!\{a, \{\!\{b, c\}\!\}''\}\!\}'_{kj}\{\!\{a, \{\!\{b, c\}\!\}''\}\!\}''_{it} \\ &= \{\!\{a, \{\!\{b, c\}\!\}'\}\!\}'_{sj}\{\!\{a, \{\!\{b, c\}\!\}'\}\!\}''_{il}\{\!\{b, c\}\!\}''_{kt} - \{\!\{a, \{\!\{c, b\}\!\}'\}\!\}'_{kj}\{\!\{a, \{\!\{c, b\}\!\}'\}\!\}''_{it}\{\!\{c, b\}\!\}''_{sl}; \\ [b_{kl}, [c_{st}, a_{ij}]] &= \{\!\{b, \{\!\{c, a\}\!\}'\}\!\}'_{il}\{\!\{b, \{\!\{c, a\}\!\}'\}\!\}''_{kt}\{\!\{c, a\}\!\}''_{sj} - \{\!\{b, \{\!\{a, c\}\!\}'\}\!\}'_{sl}\{\!\{b, \{\!\{a, c\}\!\}'\}\!\}''_{kj}\{\!\{a, c\}\!\}''_{it}; \\ [c_{st}, [a_{ij}, b_{kl}]] &= \{\!\{c, \{\!\{a, b\}\!\}'\}\!\}'_{kt}\{\!\{c, \{\!\{a, b\}\!\}'\}\!\}''_{sj}\{\!\{a, b\}\!\}_{il} - \{\!\{c, \{\!\{b, a\}\!\}'\}\!\}'_{it}\{\!\{c, \{\!\{b, a\}\!\}'\}\!\}''_{sl}\{\!\{b, a\}\!\}''_{kj}.\end{aligned}$$

Moreover,

$$\begin{aligned}\{\!\{a, b, c\}\!\} &= \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + (123) \circ \{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L + (132) \circ \{\!\{c, \{\!\{a, b\}\!\}\}\!\} \\ &= \{\!\{a, \{\!\{b, c\}\!\}'\}\!\} \otimes \{\!\{b, c\}\!\}'' + (123) \circ \{\!\{b, \{\!\{c, a\}\!\}'\}\!\} \otimes \{\!\{c, a\}\!\}'' \\ &\quad + (132) \circ \{\!\{c, \{\!\{a, b\}\!\}'\}\!\} \otimes \{\!\{a, b\}\!\}'' \\ &= \{\!\{a, \{\!\{b, c\}\!\}'\}\!\}' \otimes \{\!\{a, \{\!\{b, c\}\!\}'\}\!\}'' \otimes \{\!\{b, c\}\!\}'' \\ &\quad + \{\!\{c, a\}\!\}'' \otimes \{\!\{b, \{\!\{c, a\}\!\}'\}\!\}' \otimes \{\!\{b, \{\!\{c, a\}\!\}'\}\!\}'' \\ &\qquad + \{\!\{c, \{\!\{a, b\}\!\}'\}\!\}'' \otimes \{\!\{a, b\}\!\}'' \otimes \{\!\{c, \{\!\{a, b\}\!\}'\}\!\}'\end{aligned}$$

$$\begin{aligned}\{\!\{a,c,b\}\!\} = &\{\!\{a,\{\!\{c,b\}\!\}'\}\!\}' \otimes \{\!\{a,\{\!\{c,b\}\!\}'\}\!\}'' \otimes \{\!\{c,b\}\!\}'' \\ &+ \{\!\{b,a\}\!\}'' \otimes \{\!\{c,\{\!\{b,a\}\!\}'\}\!\}' \otimes \{\!\{c,\{\!\{b,a\}\!\}'\}\!\}'' \\ &\quad + \{\!\{b,\{\!\{a,c\}\!\}'\}\!\}'' \otimes \{\!\{a,c\}\!\}'' \otimes \{\!\{b,\{\!\{a,c\}\!\}'\}\!\}'.\end{aligned}$$

It is not difficult to see that these computations yield the desired identity. □

Note: the reason that the above is equal to 0 when A is double Poisson is as follows. Let $m : A^{\otimes^3}$ be the trilinear map sending $a \otimes b \otimes c \mapsto a_{ij}b_{kl}c_{st}$. Since linear maps always map 0 to 0, then it follows that if $a \otimes b \otimes c = 0$ in $A^{\otimes^3}$, then its image under m must also be 0. Thus, we may note that by the above, the Jacobi identity will be implied if A is double Poisson.

Example 3.11. Let $A = k[t]$ have the Double Poisson structure given by $\{\!\{t,t\}\!\} = t \otimes 1 - 1 \otimes t$. Recall that $A_n = k[t_{ij}]$ with $1 \leq i,j \leq n$. By the above, we have the following bracket on A_n:

$$[t_{ij}, t_{kl}] = \{\!\{t,t\}\!\}'_{kj}\{\!\{t,t\}\!\}''_{il} = \delta_{il}t_{kj} - \delta_{kj}t_{il}$$

(recall that the sum that should be placed to the right of the first equality has been omitted for brevity). Remember, this is exactly the bracket we defined on A_n in the second example on page 2!

Example 3.12. Let $A = k\langle x,y\rangle$, and recall that $A_n = k[x_{ij}, y_{kl}]$ for $1 \leq i,j,k,l \leq n$. Moreover, recall the double Poisson structure on this algebra that we described in Section 3.2. Obviously, it follows that

$$[x_{ij}, x_{kl}] = 0 = [y_{ij}, y_{kl}$$

for all i,j,k,l. It is not difficult to see that

$$[x_{ij}, y_{kl}] = \{\!\{x,y\}\!\}'_{ij}\{\!\{x,y\}\!\}''_{kl} = (1)_{kj}(1)_{il} = \delta_{kj}\delta_{il}$$

What about $A = k[t]$ for $\{\!\{t,t\}\!\} = t^2 \otimes t - t \otimes t^2$?

4 More on Double Brackets; Introduction to Lie Algebras

4.1 Double Brackets on Commutative (Polynomial) Algebras

Consider the algebra $A = k[y_1, \dots, y_m] \in \mathrm{ComAlg}_k$. If we want to define a double bracket $\{\!\{-,-\}\!\}$ on A, we need to enforce the restriction $\{\!\{y_1, y_1y_2\}\!\} = \{\!\{y_1, y_2y_1\}\!\}$. But by Leibniz, this implies that

$$(y_1 \otimes 1)\{\!\{y_1, y_2\}\!\} + \{\!\{y_1, y_1\}\!\}(1 \otimes y_2) = (y_2 \otimes 1)\{\!\{y_1, y_2\}\!\} + \{\!\{y_1, y_2\}\!\}(1 \otimes y_1)$$

$$(y_1 \otimes 1 - 1 \otimes y_1)\{\!\{y_1, y_2\}\!\} = (y_2 \otimes 1 - 1 \otimes y_2)\{\!\{y_1, y_1\}\!\}$$

so that $(y_2 \otimes 1 - 1 \otimes y_2)$ is a factor of $\{\!\{y_1, y_2\}\!\}$ and $(y_1 \otimes 1 - 1 \otimes y_1)$ is a factor of $\{\!\{y_2, y_1\}\!\} = -\{\!\{y_1, y_2\}\!\}^{\circ}$. In other words, we may write

$$\{\!\{y_1, y_2\}\!\} = (y_1 \otimes 1 - 1 \otimes y_1)(y_2 \otimes 1 - 1 \otimes y_2) \cdot (a \otimes b).$$

Antisymmetry and the Jacobi identity will force $\{\!\{-,-\}\!\} = 0$ by requiring the vanishing of the $a \otimes b$ term. Therefore, polynomial algebras over a field have no nontrivial double brackets.

The main idea is that double brackets don't work well in commutative algebras. They are stronger than H_0-Poisson structures, but not all H_0-Poisson structures are induced by a double bracket, as in this case.

Note that when we restrict the bracket to $A_n^{GL_n}$, which is generated by $\mathrm{tr}(a)$ for $a \in A$, we see that

$$[\mathrm{tr}(a), \mathrm{tr}(b)] = \sum_{i,j} [a_{ii}, b_{jj}] = \sum_{ij} \{\!\{a, b\}\!\}'_{ji} \{\!\{a, b\}\!\}''_{ij} = \sum_j \left(\{\!\{a, b\}\!\}' \{\!\{a, b\}\!\}'' \right)_j j = \mathrm{tr}([a, b]) \in A_n^{GL_n},$$

which is exactly the Poisson structure induced by an H_0-Poisson algebra! Thus, as noted above, double Poisson structures induce H_0-Poisson structures, but the converse is not always necessarily true.

4.2 Lie Algebras: Introduction

Definition 4.1. A Lie algebra is a k-vector space, $\mathfrak{g}$, equipped with a bilinear operation $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ called a Lie bracket satisfying

1. Anitsymmetry: $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$;
2. The Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Some examples of simple Lie algebras:

Example 4.2. Abelian Lie algebras: if $\mathfrak{g}$ is a vector space with the trivial bracket $[x, y] = 0$ for all $x, y \in \mathfrak{g}$, then it is called an abelian Lie algebra. It is obvious the above properties are satisfied trivially.

Example 4.3. Any associative algebra A may be turned into a Lie algebra by defining the Lie bracket on A in the following way: $[a, b] = ab - ba$ for all $a, b \in A$. Brackets of this form are called commutator brackets.

Example 4.4. Consider the matrix algebra $M_n(k)$ with Lie bracket as in Example 4.2. This is a Lie algebra, called the *general linear lie algebra*, and it is denoted $\mathfrak{gl}_n(k)$. Obviously it has dimension n^2.

Definition 4.5. Suppose V is a Lie algebra. A Lie subalgebra of V is a subspace U closed under the Lie bracket, i.e. $[x, y] \in U$ for all $x, y \in U$.

There are several important Lie subalgebras of $\mathfrak{gl}_n(k)$. We list them here:

Example 4.6. The special linear algebra $\mathfrak{sl}_n(k)$ is the subspace of $\mathfrak{gl}_n(k)$ of trace-zero matrices. It is easily verified that given any $x, y \in \mathfrak{sl}_n(k)$, $\operatorname{tr}([x, y]) = 0$ so that $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$. As for the dimension of $\mathfrak{sl}_n$, we see that there are no restrictions on the off-diagonal entries of an element of $\mathfrak{sl}_n$. However there are restrictions on the diagonal elements; particularly that if $\lambda_1, \ldots, \lambda_n$ are the diagonal elements of some matrix in $\mathfrak{sl}_n$, then $\lambda_1 + \ldots + \lambda_n = 0$. In particular, $\lambda_1 E_{11} + \ldots + \lambda_n E_{nn} = 0$. Now, it is not too difficult to see that $\{\lambda_1 E_{11} + \ldots + \lambda_n E_{nn} \mid \lambda_1, \ldots, \lambda_n \in k\}$ is a subspace of dimension $n-1$ (take $\{E_{11} - E_{22}, E_{22} - E_{33}, \ldots, E_{n-1,n-1} - E_{nn}\}$ to be a basis). Thus, it must be that $\dim \mathfrak{sl}_n = n^2 - n + (n-1) = n^2 - 1$.

Example 4.7. The subspace of skew-symmetric matrices also forms a Lie subalgebra of $\mathfrak{gl}_n$, which we denote by $\mathfrak{so}_n$ and is called the special orthogonal Lie algebra. Suppose $A, B \in \mathfrak{so}_n(k)$. Then

$$[A, B]^T = (AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -[A, B],$$

so $[A, B] \in \mathfrak{so}_n(k)$; $\mathfrak{so}_n$ is indeed a Lie subalgebra of $\mathfrak{gl}_n$. In fact, the skew-symmetry of the matrices in $\mathfrak{so}_n$ forces the diagonal elements of any matrix in $\mathfrak{so}_n$ to be 0, so we see that $\mathfrak{so}_n$ is a Lie subalgebra of $\mathfrak{sl}_n$. It is easy to verify that $\dim \mathfrak{so}_n = \frac{n(n-1)}{2}$. In $\mathfrak{sl}_2$, the following matrices form a basis for the entire algebra:

$$X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};\ Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Some easy computations show

$$[X, Y] = Z;\ [Z, X] = 2X;\ [Z, Y] = -2Y;$$

we may use these relations to compute the bracket of any two elements of $\mathfrak{sl}_2$.

Example 4.8. When $k = 2n$ is even, there is another Lie subalgebra of $\mathfrak{sl}_k$, called the symplectic Lie algebra. It is denoted $\mathfrak{sp}_{2n}$. Let

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Define $\mathfrak{sp}_{2n}(k) = \{X \in \mathfrak{gl}_n(k) \mid \Omega X + X^T \Omega = 0\}$. It is obvious this is a subspace of $\mathfrak{gl}_{2n}$, so we check that $\mathfrak{sp}_{2n}$ is closed under bracketing. Suppose $X, Y \in \mathfrak{sp}_{2n}$ so that $\Omega X + X^T \Omega = 0 = \Omega Y + Y^T \Omega$. Then,

$$\begin{aligned} \Omega[X, Y] + [X, Y]^T \Omega &= \Omega XY - \Omega YX + Y^T X^T \Omega - X^T Y^T \Omega \\ &= \Omega XY - \Omega YX - Y^T \Omega X + X^T \Omega Y \\ &= \Omega XY - \Omega YX + \Omega YX - \Omega XY = 0, \end{aligned}$$

so $\mathfrak{sp}_{2n}$ is indeed a Lie subalgebra of $\mathfrak{gl}_{2n}$. To find the dimension of the symplectic Lie algebra we utilize the defining relation $\Omega X + X^T \Omega = 0$. Let

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are $n \times n$ matrices. Since

$$X^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix},$$

some computation shows

$$\Omega X = \begin{bmatrix} C & D \\ -A & -B \end{bmatrix};\quad X^T \Omega = \begin{bmatrix} -C^T & A^T \\ -D^T & B^T \end{bmatrix}.$$

Because $\Omega X + X^T \Omega = 0$, then $C = C^T$, $B = B^T$, $D = -A^T$. Thus, X is completely determined by A and the upper-triangular portions of B and C. Therefore, we see that

$$\dim \mathfrak{sp}_{2n} = n^2 + n(n+1)/2 + n(n+1)/2 = n^2 + n^2 + n = 2n^2 + n.$$

Since $D = -A^T$, it is not difficult to see that $\operatorname{tr}(X) = 0$, where X is the block matrix from above. Thus, $\mathfrak{sp}_{2n}$ is a Lie subalgebra of $\mathfrak{sl}_{2n}$. Moreover, we see that by a dimension argument, $\mathfrak{sp}_{2n}$ is strictly contained in $\mathfrak{sl}_{2n}$ for $n > 1$. When $n = 1$, we see that $\mathfrak{sl}_2 = \mathfrak{sp}_2$ (again, this follows from looking at the dimensions of the respective algebras).

Now, let A be an associative k-algebra, and consider the endomorphism algebra $\mathrm{End}_k(A)$ of linear maps taking A to itself (note that in $\mathrm{End}_k(A)$, multiplication is given by composition). Recall the definition of $\mathrm{Der}(A)$, the subspace of all derivations in $\mathrm{End}_k(A)$. We show that if $f, g \in \mathrm{Der}(A)$, then $[f, g] := fg - gf$ is again a derivation, making $\mathrm{Der}(A)$ into a Lie subalgebra of $\mathrm{End}_k(A)$. Suppose $a, b \in A$. Then

$$\begin{aligned}
[f,g](ab) = (fg - gf)(ab) &= f(g(ab)) - g(f(ab)) \\
&= f(g(a)b + ag(b)) - g(f(a)b + af(b)) \\
&= g(a)f(b) + +af(g(b)) + f(a)g(b) \\
&\qquad - f(a)g(b) - g(f(a))b - a(g(f(b))) - g(a)f(b) \\
&= (f \circ g)(a)b + a(f \circ g)(b) - (g \circ f)(a)b - a(g \circ f)(b) \\
&= ([f,g])(a)b + a([f,g])(b)
\end{aligned}$$

as desired. Now, recall that the subspace of inner derivations $\mathrm{Inn}(A) \subseteq \mathrm{Der}(A)$ is given by $g_a : A \to A$ for some $a \in A$, and g_a sends $b \in A$ to $b \mapsto [a, b] = ab - ba$. It is easily verified that this is a subspace; we check that it is closed under addition: for any $x, y, b \in A$

$$(g_x + g_y)(b) = [x, b] + [y, b] = [x + y, b] = g_{x+y}(b),$$

so $x + y \in \mathrm{Inn}(A)$. Moreover, this is a Lie subalgebra of $\mathrm{Der}(A)$, since for all $g_x, g_y \in \mathrm{Inn}(A)$ and $a \in A$,

$$\begin{aligned}
[g_x, g_y](a) = (g_x g_y - g_y g_x)(a) &= g_x(g_y(a)) - g_y(g_x(a)) \\
&= [x, [y, a]] - [y, [x, a]] \\
&= [x, [y, a]] + [y, [a, x]] \\
&= -[a, [x, y]] = [[x, y], a] = g_{[x,y]}(a),
\end{aligned}$$

where the third and fifth equalities follow from the antisymmetry of the Lie bracket, and the fourth from the Jacobi identity. Thus, $g_{[x,y]} \in \mathrm{Inn}(A)$, implying $\mathrm{Inn}(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$ and hence $\mathrm{End}_k(A)$.

5 Double Brackets on $T(V)$

5.1 Homogeneous Double Brackets on $k[t]$

We begin by classifying the homogeneous double Poisson structures on $k[t]$, i.e. the double Poisson brackets for which the terms of $\{\!\{t,t\}\!\}$ all have the same degree. For example, a third degree homogeneous double Poisson structure on $k[t]$ is of the form

$$\{\!\{t,t\}\!\} = at^3 \otimes t + bt^2 \otimes t + ct \otimes t^2 + d1 \otimes t^3$$

for $a,b,c,d \in k$.

Example 5.1. Degree 0. It is not difficult to see that there are no nontrivial degree-zero homogeneous double Poisson structures on $A := k[t]$; in fact there are no nontrivial double brackets on this algebra. Suppose $\{\!\{-,-\}\!\}$ is a double bracket on A. Then, for some $a, \in k$,

$$\{\!\{t,t\}\!\} = a1 \otimes 1 = -a1 \otimes 1 = -\{\!\{t,t\}\!\}^\circ,$$

which implies $a = 0$.

Example 5.2. Degree 1. Suppose $\{\!\{t,t\}\!\} = at \otimes 1 + b1 \otimes t$ is a double Poisson structure on A. Then, $\{\!\{t,t\}\!\} = -bt \otimes 1 - a1 \otimes t = -\{\!\{t,t\}\!\}^\circ$, implying $b = -a$. It is not difficult to check that $\{\!\{t,t\}\!\} = at \otimes 1 - a1 \otimes t$ satisfies the Double Jacobi identity for any $a \in k$, so $\{\!\{t,t\}\!\} = t \otimes 1 - 1 \otimes t$ is the only degree-one double Poisson structure on A up to rescaling.

Example 5.3. Degree 2. It was checked the previous meeting that there are no nontivial, degree-two double Poisson structures on A.

Example 5.4. Degree 3. This was assigned as homework. We show that the only degree-three homogeneous, nontrivial double Poisson bracket on A is $\{\!\{t,t\}\!\} = t^2 \otimes t - t \otimes t^2$ up to multiplication by a scalar. Suppose $\{\!\{t,t\}\!\}$ is a double Poisson bracket on A of degree 3. Then, the bracket must have the following form

$$\{\!\{t,t\}\!\} := at^3 \otimes 1 + bt^2 \otimes t + ct \otimes t^2 + d1 \otimes t^3.$$

The antisymmetry of the double bracket implies that

$$\{\!\{t,t\}\!\} = -(dt^3 \otimes 1 + ct^2 \otimes t + bt \otimes t^2 + a1 \otimes t^3);$$

combining this with the above implies $d = -a$ and $c = -b$, so

$$\{\!\{t,t\}\!\} = at^3 \otimes 1 + bt^2 \otimes t - bt \otimes t^2 - a1 \otimes t^3.$$

Now, by assumption, this bracket satisfies the double Jacobi identity, so we plug in, simplify, and record what restrictions (if any) there are on a and b. It will be helpful to know the following: $\{\!\{t,t^2\}\!\}$ and $\{\!\{t,t^3\}\!\}$. By repeated application of the Leibniz rule, we see that

$$\begin{aligned}
\{\!\{t,t^2\}\!\} &= \{\!\{t,t\}\!\}t + t\{\!\{t,t\}\!\} \\
&= at^3 \otimes t + bt^2 \otimes t^2 - bt \otimes t^3 - a1 \otimes t^4 \\
&\qquad + at^4 \otimes 1 + bt^3 \otimes t - bt^2 \otimes t^2 - at \otimes t^3 \\
&= at^4 \otimes 1 - a1 \otimes t^4 + (a+b)t^3 \otimes t - (a+b)t \otimes t^3;
\end{aligned}$$

$$\begin{aligned}
\{\!\{t,t^3\}\!\} &= \{\!\{t,t\}\!\}t^2 + t\{\!\{t,t^2\}\!\} \\
&= at^3 \otimes t^2 + bt^2 \otimes t^3 - bt \otimes t^4 - a1 \otimes t^5 \\
&\qquad + at^5 \otimes 1 - at \otimes t^4 + (a+b)t^4 \otimes t - (a+b)t^2 \otimes t^3 \\
&= at^5 \otimes 1 - a1 \otimes t^5 + (a+b)t^4 \otimes t - (a+b)t \otimes t^4 + at^3 \otimes t^2 - at^2 \otimes t^3.
\end{aligned}$$

We are now ready to compute $\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$:

$$\begin{aligned}\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L &= a\{\!\{t, t^3\otimes 1\}\!\}_L + b\{\!\{t, t^2\otimes t\}\!\}_L - b\{\!\{t, t\otimes t^2\}\!\}_L - a\{\!\{t, 1\otimes t^3\}\!\}_L \\ &= a\{\!\{t,t^3\}\!\}\otimes 1 + b\{\!\{t,t^2\}\!\}\otimes t - b\{\!\{t,t\}\!\}\otimes t^2 \\ &= a^2t^5\otimes 1\otimes 1 - a^2 1\otimes t^5\otimes 1 + a(a+b)t^4\otimes t\otimes 1 - a(a+b)t\otimes t^4\otimes 1 \\ &\quad + a^2t^3\otimes t^2\otimes 1 - a^2t^2\otimes t^3\otimes 1 + abt^4\otimes 1\otimes t - ab1\otimes t^4\otimes t \\ &\quad + b(a+b)t^3\otimes t\otimes t - b(a+b)t\otimes t^3\otimes t - abt^3\otimes 1\otimes t^2 \\ &\quad - b^2t^2\otimes t\otimes t^2 + b^2t\otimes t^2\otimes t^2 + ab1\otimes t^3\otimes t^2.\end{aligned}$$

Now, we group these terms by their orbits under cyclic permutations of the tensors. For example, $a^2t^3\otimes t^2\otimes 1$ and $ab1\otimes t^3\otimes t^2$ would be in one group, whereas $-a^2t^2\otimes t^3\otimes 1$ and $-abt^3\otimes 1\otimes t^2$ would be an another. Recall that the other terms of the double Jacobi identity are $(123)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$ and $(132)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$; these are easily calculated from the above. Add the corresponding terms from $(123)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$ and $(132)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$ to each grouping, so if G is an orbit, we may set $\sum_{g\in G} g = 0$, since we have assumed the double Jacobi identity. To make this concrete, we have, as an example,

$$\begin{aligned}0 =& abt^4\otimes 1\otimes t - a(a+b)t\otimes t^4\otimes 1 \\ &+ abt\otimes t^4\otimes 1 - a(a+b)1\otimes t\otimes t^4 \\ &+ ab1\otimes t\otimes t^4 - a(a+b)t^4\otimes 1\otimes t \\ =& -a^2(t\otimes t^4\otimes 1 + 1\otimes t\otimes t^4 + t^4\otimes 1\otimes t).\end{aligned}$$

Clearly, this implies $a = 0$, so

$$\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L = b^2t^3\otimes t\otimes t - b^2t\otimes t^3\otimes t - b^2t^2\otimes t\otimes t^2 + b^2t\otimes t^2\otimes t^2.$$

Combining this with $(123)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$ and $(132)\circ\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L$, we see that b can be any element in k. Thus, $\{\!\{t,t\}\!\} = bt^2\otimes t - bt\otimes t^2$. Therefore, up to rescaling $t^2\otimes t - t\otimes t^2$ is the only degree-three, nontivial homogeneous double Poisson bracket on $k[t]$.

Theorem 5.5. *There are no nontrivial homogeneous-of-degree-n double Poisson brackets on $k[t]$ for $n \geq 4$.*

Proof. We apply an approach similar to our work on the degree 3 case. Note that $\{\!\{t,t\}\!\}$ is of the form

$$\{\!\{t,t\}\!\} = \sum_{\frac{n}{2}<i\leq n} a_i(t^{n-i}\otimes t^i - t^i\otimes t^{n-i})$$

for $a_i \in k$ for all i. We outline this proof for the simple cases, mainly to illustrate that it is doable for the most general case. So first assume $\{\!\{t,t\}\!\} = a(t^n\otimes 1 - 1\otimes t^n)$ and then that $\{\!\{t,t\}\!\} = b(t^m\otimes t^{m-1} - t^{m-1}\otimes t^m)$. We see that

$$\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L = a\{\!\{t,t^n\}\!\}\otimes 1$$

Expanding everything out using the Leibniz rule, we see one of the terms in the expansion is $at^{n-2}\{\!\{t,t\}\!\}t\otimes 1$. From this, we select the term $at^{2n-2}\otimes t\otimes 1$ and group the elements in the above that are in the orbit of $at^{2n-2}\otimes t\otimes 1$ under cyclic permutation of the tensors. From here, the process is almost exactly the same as in Example 5.4. The other case is a bit more complicated. If $\{\!\{t,t\}\!\} = b(t^m\otimes t^{m-1} - t^{m-1}\otimes t^m)$, then it is not difficult to see that

$$\{\!\{t, \{\!\{t,t\}\!\}\}\!\}_L = b(\{\!\{t,t^m\}\!\}\otimes t^{m-1} - b\{\!\{t,t^{m-1}\}\!\}\otimes t^m).$$

We expand out the first term of the difference:

$$\begin{aligned}b\{\!\{t,t^m\}\!\} &= b(\{\!\{t,t\}\!\}t^{m-1} + t\{\!\{t,t\}\!\}t^{m-2} + \ldots + t^{m-1}\{\!\{t,t\}\!\}) \\ &= bt^m\otimes t^{2m-2} - bt^{m-1}\otimes t^{2m-1} \\ &\quad + bt^{m+1}\otimes t^{2m-3} - bt^m\otimes t^{2m-2} \\ &\quad + bt^{m+2}\otimes t^{2m-4} - bt^{m+1}\otimes t^{2m-3} \\ &\quad + \ldots + bt^{2m-1}\otimes t^{m-1} - b^{2m-2}\otimes t^{2m-1} \\ &= bt^{2m-1}\otimes t^{m-1} - bt^{m-1}\otimes t^{2m-1},\end{aligned}$$

so it follows that

$$b\{\!\{t, t^m\}\!\} \otimes t^{m-1} = bt^{2m-1} \otimes t^{m-1} \otimes t^{m-1} - bt^{m-1} \otimes t^{2m-1} \otimes t^{m-1}.$$

The other term of the difference can be computed similarly; it is

$$-b\{\!\{t, t^{m-1}\}\!\} \otimes t^m = -bt^{2m-2} \otimes t^{m-1} \otimes t^m + bt^{m-1} \otimes t^{2m-2} \otimes t^m.$$

Therefore,

$$\begin{aligned}\{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L = bt^{2m-1} &\otimes t^{m-1} \otimes t^{m-1} - bt^{m-1} \otimes t^{2m-1} \otimes t^{m-1} \\ &- bt^{2m-2} \otimes t^{m-1} \otimes t^m + bt^{m-1} \otimes t^{2m-2} \otimes t^m.\end{aligned}$$

From here, $(123) \circ \{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L$ and $(132) \circ \{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L$ are easily computed. It is not too difficult to see that the double Jacobi identity is satisfied only when $m = 1$ or $m = 2$, which is compatible with our results above. □

5.2 Nonhomogeneous Double Brackets on $k[t]$

Let $\{\!\{-, -\}\!\}$ be a nonhomogeneous double Poisson bracket on $k[t]$, so that $\{\!\{-, -\}\!\}$ is some linear combination of homogeneous Poisson structures on $k[t]$. In other words,

$$\{\!\{t, t\}\!\} = a_1\{\!\{t, t\}\!\}_1 + a_2\{\!\{t, t\}\!\}_2 + \ldots + a_n\{\!\{t, t\}\!\}_n,$$

where $\{\!\{t, t\}\!\}_i$ is a degree i homogeneous bracket. Then,

$$\{\!\{t, \{\!\{t, t\}\!\}\}\!\}_L = \sum_{1 \le i,j \le n} a_i a_j \{\!\{t, \{\!\{t, t\}\!\}'_j\}\!\}_i \otimes \{\!\{t, t\}\!\}''_j.$$

Note that $\{\!\{t, \{\!\{t, t\}\!\}'_j\}\!\}_i \otimes \{\!\{t, t\}\!\}''_j$ has degree $i + j - 1$, so the term with the lowest degree is when $i = j = 1$; the term with the highest is when $i = j = n$. Since we are assuming $\{\!\{-, -\}\!\}$ satisfies the double Jacobi identity, elements of the same degree must cancel, since permuting the tensors cyclically will not alter their degree. Thus, it follows that the brackets $\{\!\{-, -\}\!\}_1$ and $\{\!\{-, -\}\!\}_n$ are double Poisson, which tells us that $n \le 3$ by the last section, since $\{\!\{-, -\}\!\}_i$ is double Poisson for $i = 1$ and $i = 3$. Therefore,

$$\{\!\{t, t\}\!\} = a(t \otimes 1 - 1 \otimes t) + b(t^2 \otimes t - t \otimes t^2) + c(t^2 \otimes t - t \otimes t^2),$$

and $\{\!\{t, \{\!\{t, t\}\!\}\}\!\}$ has terms of degree 1, 2, 3, 4, and 5. We already know that the degree 1 and 5 terms satisfy the double Jacobi identity and leave no restrictions on a, b, or c. For the degree 2 terms, we have $i = 1$ and $j = 2$, and vice versa. When $i = 1$, $j = 2$, then we get

$$ab\{\!\{t, t^2\}\!\}_1 \otimes 1 = ab(t^2 \otimes 1 \otimes 1 - t \otimes t \otimes 1 + t \otimes t \otimes 1 - 1 \otimes t^2 \otimes 1),$$

and when $i = 2$, $j = 1$, we get

$$ab\{\!\{t, t\}\!\}_2 \otimes 1\_2 \otimes 1 = ab(t^2 \otimes 1 \otimes 1 - 1 \otimes t^2 \otimes 1).$$

It is not difficult to see that all of these degree 2 terms will cancel after adding the requisite cyclic permutations of the tensors. For the degree 4 terms, the possible i and j are 2 and 3. When $i = 3$ and $j = 2$, we have

$$bc(\{\!\{t, t^2\}\!\}_2 \otimes t - \{\!\{t, t\}\!\}_2 \otimes t^2) = bc(t^3 \otimes 1 \otimes t - t \otimes t^2 \otimes t + t^2 \otimes t \otimes t - 1 \otimes t^3 \otimes t - t^2 \otimes 1 \otimes t^2 + 1 \otimes t^2 \otimes t^2).$$

When $i = 2$ and $j = 3$, then we have

$$bc(\{\!\{t, t^2\}\!\}_3 \otimes 1) = bc(t^3 \otimes t \otimes 1 - t^2 \otimes t^2 \otimes 1 + t^2 \otimes t^2 \otimes 1 - t \otimes t^3 \otimes 1) = bc(t^3 \otimes t \otimes 1 - t \otimes t^3 \otimes 1).$$

After adding the cyclically-permuted tensors, we see that all of the degree 4 terms cancel, again leaving us with no restrictions on a, b, or c. The last and most time consuming case to check are the degree 3 tensors. There are three options for i and j: $i = 1$, $j = 3$; $i = 3$, $j = 1$; $i = 2 = j$. When $i = 1$ and $j = 3$, the computation is simple:

$$ac\{\!\{t,t\}\!\}_3 \otimes 1 = ac(t^2 \otimes t \otimes 1 - t \otimes t^2 \otimes 1).$$

For $i = 3$ and $j = 1$, we see that

$$\begin{aligned} ac(\{\!\{t,t^2\}\!\}_1 \otimes t - \{\!\{t,t\}\!\}_1 \otimes t^2) &= ac(t \otimes t \otimes t - 1 \otimes t^2 \otimes t + t^2 \otimes 1 \otimes t \\ &\qquad - t \otimes t \otimes t - t \otimes 1 \otimes t^2 + 1 \otimes t \otimes t^2) \\ &= ac(-1 \otimes t^2 \otimes t + t^2 \otimes 1 \otimes t - t \otimes 1 \otimes t^2 + 1 \otimes t \otimes t^2); \end{aligned}$$

for $i = j = 2$, we have

$$b^2(\{\!\{t,t^2\}\!\}_2 \otimes 1) = b^2(t^2 \otimes t \otimes 1 - 1 \otimes t^3 \otimes 1 + t^3 \otimes 1 \otimes 1 - t \otimes t^2 \otimes 1).$$

Note that the middle two terms in the above expression will cancel after permuting. Thus, we may now follow the grouping procedure we applied previously, in which we group the tensors by their orbits under cyclic permutations. We deal with one of these groups; the other is treated similarly. Specifically, we have

$$\begin{aligned} -act \otimes t^2 \otimes 1 - b^2 t \otimes t^2 \otimes 1 + act^2 \otimes 1 \otimes t + ac1 \otimes t \otimes t^2 \\ = act^2 \otimes 1 \otimes t + ac1 \otimes t \otimes t^2 - (ac + b^2)t \otimes t^2 \otimes 1, \end{aligned}$$

so

$$\begin{aligned} 0 &= act^2 \otimes 1 \otimes t + ac1 \otimes t \otimes t^2 - (ac + b^2)t \otimes t^2 \otimes 1 \\ &\quad + act \otimes t^2 \otimes 1 + act^2 \otimes 1 \otimes t - (ac + b^2)1 \otimes t \otimes t^2 \\ &\qquad + ac1 \otimes t \otimes t^2 + act \otimes t^2 \otimes 1 - (ac + b^2)t^2 \otimes 1 \otimes t \\ &= ac1 \otimes t \otimes t^2 - b^2 t \otimes t^2 \otimes 1 + act^2 \otimes 1 \otimes t - b^2 1 \otimes t \otimes t^2 + act \otimes t^2 \otimes 1 - b^2 t^2 \otimes 1 \otimes t \\ &= (ac - b^2)t^2 \otimes 1 \otimes t + (ac - b^2)1 \otimes t \otimes t^2 + (ac - b^2)t \times t^2 \otimes 1, \end{aligned}$$

which obviously implies $ac - b^2 = 0$. Thus, any nonhomogeneous double Poisson bracket on $k[t]$ must be of the form

$$\{\!\{t,t\}\!\} = a(t \otimes 1 - 1 \otimes t) + b(t^2 \otimes t - t \otimes t^2) + c(t^2 \otimes t - t \otimes t^2)$$

where $ac = b^2$. For a clearer and more complete version of this proof, see the paper by Powell.

5.3 Homogeneous Double Poisson Brackets on $T(V)$

Notably, all of our work above has been restricted to $k[t]$, which, of course, begs the question: what about algebras with multiple generators? We consider the tensor algebra $T(V)$, with $x_1, \ldots, x_n$ as a basis. This gets complicated pretty quickly, so we focus only on the degree 0 and degree 1 cases. Suppose the double bracket defined on $T(V)$ is homogeneous of degree 0. Then,

$$\{\!\{x_i, x_j\}\!\} = a_{ij} 1 \otimes 1 = -a_{ji} 1 \otimes 1 = -\{\!\{x_i, x_j\}\!\}^\circ, \tag{2}$$

so we see that any homogeneous double Poisson bracket of degree 0 is completely determined by a skew-symmetric matrix (note that the double Jacobi identity is satisfied trivially). If we consider the classic example when $T(V) = k\langle x, y\rangle$, $\{\!\{x,x\}\!\} = 0 = \{\!\{y,y\}\!\}$, and $\{\!\{x,y\}\!\} = 1 \otimes 1$, it is not difficult to see that the a_{ij}'s given in (2), in this case, are determined by

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The degree 1 case is far more complex. Here, we have

$$\{\!\{x_i, x_j\}\!\} = \sum_{k=1}^{n} a_{ij}^k x_k \otimes 1 - b_{ij}^k 1 \otimes x_k = \sum_{k=1}^{n} -b_{ji}^k x_k \otimes 1 + a_{ji}^k 1 \otimes x_k = -\{\!\{x_j, x_i\}\!\}^\circ,$$

so

$$\{\!\{x_i, x_j\}\!\} = \sum_{k=1}^{n} a_{ij}^k x_k \otimes 1 - a_{ji}^k 1 \otimes x_k.$$

Now, assume the bracket satisfies the double Jacobi identity so that it is double Poisson. Thus, we have, for all i, j, k,

$$\{\!\{x_i, \{\!\{x_j, x_k\}\!\}\}\!\}_L + (123) \circ \{\!\{x_j, \{\!\{x_k, x_i\}\!\}\}\!\}_L + (132) \circ \{\!\{x_k, \{\!\{x_i, x_j\}\!\}\}\!\}_L = 0.$$

Note that

$$\{\!\{x_i, \{\!\{x_j, x_k\}\!\}\}\!\}_L = \{\!\{x_i, \sum_s a_{jk}^s x_2 \otimes 1 - a_{kj}^s 1 \otimes x_s\}\!\}_L = \sum_s a_{jk}^s \{\!\{x_i, x_s\}\!\} \otimes 1;$$

similarly,

$$(123) \circ \{\!\{x_j, \{\!\{x_k, x_i\}\!\}\}\!\}_L = \sum_s a_{ki}^2 \{\!\{x_j, x_s\}\!\} \otimes 1,$$

and

$$(132) \circ \{\!\{x_k, \{\!\{x_i, x_j\}\!\}\}\!\}_L = \sum_s a_{ij}^s \{\!\{x_k, x_s\}\!\} \otimes 1.$$

Therefore, we have

$$\begin{aligned} 0 = \sum_{s,t} & a_{jk}^s a_{is}^t x_t \otimes 1 \otimes 1 - a_{jk}^s a_{si}^t 1 \otimes x_t \otimes 1 \\ & + a_{ki}^s a_{js}^t 1 \otimes x_t \otimes 1 - a_{ki}^s a_{sj}^t 1 \otimes 1 \otimes x_t \\ & + a_{ij}^s a_{ks}^t 1 \otimes 1 \otimes x_t - a_{ij}^s a_{sk}^t x_t \otimes 1 \otimes 1. \end{aligned}$$

This induces the following relations:

$$\sum_{s,t} a_{is}^t a_{jk}^s = \sum_{s,t} a_{ij}^s a_{sk}^t; \quad \sum_{s,t} a_{ks}^t a_{ij}^s = \sum_{s,t} a_{ki}^s a_{sj}^t;$$

and

$$\sum_{s,t} a_{is}^t a_{jk}^s = \sum_{s,t} a_{ij}^s a_{sk}^t.$$

If we consider s as a function of i, j, k and t as a function of i, j, k, s, then we see that for each relation we get an equivalent diagram. For example, the last relation implies

$$\begin{tikzcd}
j \arrow[dr] & k \arrow[d] & i \arrow[dd] & & i \arrow[dr] & j \arrow[d] & k \arrow[dd] \\
& s \arrow[dr] & & = & & s \arrow[dr] & \\
& & t & & & & t
\end{tikzcd}$$

(In other words... "Doing" j and k first, and then i is the same as "doing" i and j first and then k.) Thus if were to define an algebra structure on $T(V)$ by $x_i * x_j = \sum_s a_{ij}^s x_s$, it would clearly be associative, i.e. $x_i * (x_j * x_k) = (x_i * x_j) * x_k$. Therefore, we have a bijection between the set of linear double Poisson structures on $T(V)$ and the set of nonunital associative algebra structures on V.

6 The Relationship Between Linear Double Poisson Structures on $T(V)$ and Associative Algebra Structures on V

6.1 Observations on $k[x]$

We first note that $k[x] \otimes k[y] \cong k[x, y]$. In fact, for any two polynomial algebras over k on n and m generators respectively, their tensor product is isomorphic, as k-algebras, to the polynomial algebra on $m+n$ generators. However, as we shall see, the commutativity is vital; this fact will not hold when the algebras are noncommutative, i.e. $k\langle x_1, y_1\rangle \otimes k\langle x_2, y_2\rangle \ncong k\langle x_1, y_1, x_2, y_2\rangle$. We prove this by applying the following universal property of tensor algebras. Let A, B, C be commutative k-algebras. Then, each algebra contains a copy of k, since for all $\lambda \in k$, $\lambda 1_X \in X$ for $X \in \{A, B\}$. Moreover, for each of A and B, there is a natural "inclusion" map $i_X : X \to A \otimes B$ for $X \in \{A, B\}$, where $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$. Now, given any two algebra homomorphisms $f : A \to C$ and $g : B \to C$, we let $f \otimes g$ be the unique algebra homomorphism from $A \otimes B \to C$ such that the following diagram commutes:

$$\begin{tikzcd}
k \arrow[r, "\lambda 1_B"] \arrow[d, "\lambda 1_A"'] & B \arrow[d, "i_B"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "i_A"] \arrow[drr, bend right, "f"'] & A \otimes B \arrow[dr, dotted, "f\otimes g"] & \\
 & & C
\end{tikzcd}$$

Since we are working in the category of algebras, here we may assume all maps are algebra homomorphisms. It is not difficult to see that letting $f \otimes g(a \otimes b) = f(a)g(b)$ gives us what we want; here $A \otimes B$ is the coproduct of the diagram. We check that $f \otimes g$ preserves multiplication: for any $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes B$,

$$\begin{aligned}
f \otimes g(a_1 \otimes b_1 \cdot a_2 \otimes b_2) = f \otimes g(a_1 a_2 \otimes b_1 b_2) &= f(a_1 a_2) g(b_1 b_2) \\
&= f(a_1) f(a_2) g(b_1) g(b_2) \\
&= f \otimes g(a_1 \otimes b_1) \cdot f \otimes g(a_2 \otimes b_2)
\end{aligned}$$

as desired, noting here the importance of commutativity in C. This can be applied to $k[x]$ and $k[y]$ by setting $A = k[x]$, $B = k[y]$, and $C = k[x, y]$, giving us the following diagram:

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[y] \arrow[d] \arrow[ddr, bend left] & \\
k[x] \arrow[r] \arrow[drr, bend right] & k[x] \otimes k[y] \arrow[dr, dotted] & \\
 & & k[x, y]
\end{tikzcd}$$

6.2 Linear Double Poissons on $T(V)$

We begin by recalling the fact that there exists a bijection between

$$\{\text{Linear Double Poissons on } T(V)\} \longleftrightarrow \{\text{(nonunital) associative algebra structures on } V\}.$$

Supposing $V = \operatorname{span}\{x_1, \ldots, x_n\}$, any double Poisson structure on $T(V)$ will be completely determined by how it acts on generators. In other words, we have

$$\{\!\{x_i, x_j\}\!\} := \sum_k a_{ij}^k x_k \otimes 1 + b_{ij}^k 1 \otimes x_k = -\{\!\{x_j, x_i\}\!\}^{\circ},$$

so $b_{ij}^k = -a_{ji}^k$. Hence, we have

$$\{\!\{x_i, x_j\}\!\} = \sum_k a_{ij}^k x_k \otimes 1 - a_{ji}^k 1 \otimes x_k,$$

so the double bracket is completely determined by the a_{ij}^k. Since we are assuming the bracket to be double Poisson, it satisfies the Double Jacobi identity, which when fully worked out, gives the following relations:

$$\sum_{s,t} a_{is}^t a_{jk}^s = \sum_{s,t} a_{ij}^s a_{sk}^t; \quad \sum_{s,t} a_{ks}^t a_{ij}^s = \sum_{s,t} a_{ki}^s a_{sj}^t;$$

and

$$\sum_{s,t} a_{is}^t a_{jk}^s = \sum_{s,t} a_{ij}^s a_{sk}^t$$

(note that these relations are not really distinct—they are just cyclic permutations of each other; we can reindex so that they are the same). Thinking of the bottom two indices as inputs; the top as an output, we define a multiplication on V by

$$x_i * x_j = \sum_s a_{ij}^s x_s.$$

It is not difficult to see that this multiplication is associative; this is induced by the relations established above:

$$\begin{aligned} x_i * (x_j * x_k) = x_i * \sum_s a_{jk}^s = \sum_s a_{jk}^s x_i * x_s &= \sum_s a_{jk}^s \sum_t a_{is}^t x_t = \sum_{s,t} a_{is}^t a_{jk}^s x_t \\ &= \sum_{s,t} a_{ij}^s a_{sk}^t x_t = \sum_s a_{ij}^s x_s * x_k \\ &= (x_i * x_j) * x_k. \end{aligned}$$

Now, it can be shown that this construction may be done backwards: Starting with an associative (nonunital) multiplication on V, we can find a_{ij}^s, $1 \le s \le n$ for every x_i, x_j in the basis of V. Then, we define a double bracket on $\{x_1, \dots, x_n\}$ and extend to the entirety of $T(V)$. It is not difficult to see that this recovers a double Poisson structure on $T(V)$.

6.3 A Lie-algebraic Analog

Recall the definition of a Lie algebra (see Section 4 for this material). Suppose $\mathfrak{g}$ is a Lie algebra; for a moment, forget about its Lie bracket. Then, we define Sym($\mathfrak{g}$) to be the commutative free algebra generated by $\mathfrak{g}$. We have the following property:

Proposition 6.1. *Sym*($\mathfrak{g}$) *is a Poisson algebra.*

Proof. This is straightforward. Let $\{-,-\}$ be defined by $\{x,y\} = [x,y]$ for all $x, y \in \mathfrak{g}$, where $[-,-]$ is the Lie bracket on $\mathfrak{g}$. Now extend using the Leibniz rule so that $\{\!\{-,-\}\!\}$ is defined for all of Sym($\mathfrak{g}$). The other relevant axioms are satisfied trivially, so Sym($\mathfrak{g}$) is indeed a Poisson algebra as desired. □

Conversely, any linear Poisson bracket on Sym(V) for a vector space V may be restricted to obtain a Lie bracket on V. In other words,

$$\{x_i, x_j\} = \sum_k a_{ij}^k x_k$$

gives us a Lie bracket on V by restricting $\{x_i, x_j\}$ to $\{x_1, \dots, x_n\}$. Thus, we get the following bijective correspondence:

$$\{\text{Linear Poissons on Sym}(V)\} \longleftrightarrow \{\text{Lie algebra structures on } V\}.$$

This classical result is well-known; this result and the one relating linear double Poissons to associative algebra structures are clearly very similar. So, the natural question is: Is there a relation between the above result and the one we recalled in the previous section? The relationship is illustrated by the below diagram:

$$\begin{tikzcd}
\{\text{Linear Double Poissons on } T(V)\} \arrow[r, "\{-,-\}_{T(V)_{\natural\natural}}"'] \arrow[d, leftrightarrow] & \{\text{Linear Poissons on } \mathrm{Sym}(V)\} \arrow[d, leftrightarrow] \\
\{\text{Assoc. Algebra Structures on } V\} \arrow[r, "xy-yx"'] & \{\text{Lie Algebra Structures on } V\}
\end{tikzcd}$$

We note that the bijections in the left and right "columns" have already been shown. Moreover, the lowermost arrow results from the fact that any associative algebra becomes a Lie algebra when equpped with the commutator bracket. Finally, we note that if a bracket on $T(V)$ is Double Poisson, then a Linear Poisson structure is induced on $\mathrm{Sym}(V)$. Here, $\mathrm{Sym}(V)$ can be constructed by taking the quotient of T(V) be the ideal generated by elements of the form $x \otimes y - y \otimes x$. That is, $\mathrm{Sym}(V) = T(V)_{\natural\natural}$. Thus, the Linear Poisson structure that the Double Poisson Bracket on T(V) induces on T(V) can also be applied to $T(V)_{\natural\natural}$, giving us the uppermost arrow in the above diagram. We therefore have the following implications:

$$\begin{tikzcd}
\{\text{Linear Double Poissons on } T(V) = A\} \arrow[r, "\{-,-\}_{T(V)_{\natural\natural}}"'] \arrow[d, leftrightarrow] & \{\text{Linear Poissons on } A_n\} \arrow[d, leftrightarrow] \\
\{\text{Associative Algebra Structures on } V\} \arrow[r, dashed, "?"'] & \{\text{Lie Algebra Structures on } V_n\}
\end{tikzcd}$$

In case $n = 1$, it is clear that for an (not necessarily unital) associative algebra structure on V, a Lie algebra structure on V_1 can be induced by defining the Lie Bracket as the commutator. It has been left as an exercise to determine if there is a systematic way for determining how an associative algebra structure on V gives a Lie Algebra structure on V_n for $n > 1$. In other words, find fourth arrow marked with ? above. (Note here that this is an open ended question—there may or may not be an interesting relation.)

6.4 Graded Algebras

Definition 6.2. A graded vector space is a k-vector space V such that

$$V = \bigoplus_{i \in I} V_i,$$

where I is some index set (usually $\mathbb{Z}$ or $\mathbb{N}$; if $I = \mathbb{N}$, we say V is nonnegative graded) and V_i is a k-vector space for all $i \in I$. So, if $v \in V$, $v = \sum_i v_i$, $v_i \in V_i$ for all i. We say that the degree of v_i, denoted $|v_i|$, is i.

Definition 6.3. A graded algebra A is exactly what one would think. As a vector space, A is graded; the grading is such that for all $a_i \in A_i$ and $a_j \in A_j$, $a_i a_j \in A_{i+j}$.

Example 6.4. $k[x]$ is a nonnegative graded algebra, with its grading being the degrees of the standard basis vectors $\{1, x, x^2, \ldots\}$. Likewise, $k[x, y] \cong k[x] \otimes k[y]$ is a nonnegative graded algebra, where the homogeneous degree determines the grading.

It is not difficult to see that a grading on V gives a grading of $T(V)$. Moreover, if V and W are graded algebras, there is a grading on $V \otimes W$ given by

$$V \otimes W = \bigoplus_k \Big(\bigoplus_{i+j=k} V_i \otimes W_j \Big).$$

For a double bracket of degree n, $\{\!\!\{-,-\}\!\!\}$, we have $a \otimes b \mapsto \{\!\!\{a, b\}\!\!\}$. Clearly, $a \otimes b$ has degree $|a| + |b|$; the degree of $\{\!\!\{a, b\}\!\!\}$ is $|a| + |b| + n$. See Section 7.1 for more on graded algebras.

6.5 Products and Coproducts

If V is a graded vector space, so that $V = \bigoplus_i V_i$, we have two sorts of linear dual spaces of V: V^* and $\bigoplus_i V_i^*$. In the second "dual space," the elements are of the form $\sum_i f_i$ for $f_i \in V_i^*$ for finitely many nonzero i, whereas any linear functional $f \in V^*$ can be nonzero on all $v_i \in V_i$. In fact, we have that $\bigoplus_i V_i^* \subseteq \prod_i V_i^* = V^*$, so $\bigoplus_i V_i^*$ is a graded linear dual space. In terms of category theory, we see that the direct sum is a colimit, whereas the direct product is a limit. They can be characterized by the following commutative diagrams:

$$\begin{tikzcd} W \arrow[r, "f_i"] \arrow[d, dashed, "F"'] & V_i \\ \prod_i V_i \arrow[ur, "\pi_i"'] & \end{tikzcd} \qquad \begin{tikzcd} W & V_i \arrow[l, "g_k"'] \arrow[dl, "i_k"] \\ \bigoplus_k V_k \arrow[u, dashed, "G"] & \end{tikzcd}$$

In the first diagram, W is a k-vector space, and each f_i is a map $W \to V_i$. The π_i's are the familiar projection maps. Now, F is the map induced by the diagram, i.e. the map that makes the diagram commute. It is not difficult to see that $F(w) = (f_1(w), f_2(w), \ldots, f_n(w), \ldots)$. As for the second diagram, we have $g_k : V_k \to W$ for all k, and $i_k : V_k \to \bigoplus_k V_k$ being the natural inclusion maps $v_k \mapsto (0, \ldots, v_k, 0, \ldots)$, where v_k is in the kth spot. Similarly, G is the linear map making the diagram commute; clearly $G(v_1, \ldots, v_k, 0, \ldots) = g(v_1) + \ldots + g(v_k)$. Note here that we have avoided dealing with any tricky business involving infinite sums. As alluded to earlier, in this case $\prod_i V_i$ and $\bigoplus_k V_k$ are, respectively, examples of a limit and colimit, which are category-theoretic notions. Specifically, they are a product and coproduct (respectively). We work another easy example of this, this time in the category of sets.

Example 6.5. Let A, B, C, D be sets, and let $f_X : C \to X$ be maps for $X \in \{A, B\}$, and let π_A, π_B be the natural projection maps from $A \times B$ to A and B respectively. Then there exists a unique map $F : C \to A \times B$ which makes the top half of the diagram below commute:

$$\begin{tikzcd} & C \arrow[d, dashed, "F"] \arrow[ddl, bend right, "f_A"'] \arrow[ddr, bend left, "f_B"] & \\ & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A \arrow[dr, "i_A"] \arrow[ddr, bend right, "g_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "g_B"] \\ & A \coprod B \arrow[d, dashed, "G"'] & \\ & D & \end{tikzcd}$$

Here, $A \times B$ is called the product of A and B, and is another example of a limit. Now, we deal with the bottom half of the diagram. If $g_X : X \to D$ are maps, where $X \in \{A, B\}$, and if $i_X : X \to A \coprod B$ is the natural inlcusion map, then $G : A \coprod B \to D$ is the unique map that makes the bottom half of the diagram commute. Then, $A \coprod B$ is called the coproduct of A and B, and is an example of a colimit. Also, note that $A \coprod B$ is, here, the disjoint union of A and B.

7 Lie Brackets on V_n and More on Graded Algebras

7.1 Graded Algebras Continued

Suppose we have a graded algebra A. Note that its multiplication must be of degree 0: for $a_n \in A_n$, $a_m \in A_m$, we have, by definition, $a_n a_m \in A_{n+m}$. If A and A' are graded k-algebras, and $f : A \to A'$ an algebra homomorphism, then for all $a, b \in A$, we have $f(ab) = f(a)f(b)$. Recall that if $|f|$, the degree of fm is m, then this means that $f(A_n) \subseteq A'_{n+m}$. Hence, the lefthand side $f(ab) = f(a)f(b)$ has degree $|a| + |b| + |f|$ whereas the righthand side has degree $|a| + |b| + 2|f|$. The degrees must be equal, so $|f| = 0$ (note that $\mathrm{char}(k) \neq 2$). Thus, if given a graded vector space V, in order to define a multiplication structure on V, i.e. a map $m : V \otimes V \to V$, we must ensure that it is of degree 0. Recall that if V and W are graded vector spaces, then $V \otimes W$ is also a graded vector space. Thus, if given a bilinear map $f : V \otimes V \to V$ of degree n (i.e. $f((V \otimes V)_m) \subseteq V_{n+m}$), we need to artificially adjust the degree of V so that f becomes a multiplication on V. To do this, note that for all $v \otimes w \in V \otimes V$, $|f(v \otimes w)| = n + |v| + |w|$. Then, it is not difficult to see that adding n to both sides gives

$$n + |f(v \otimes w)| = (|v| + n) + (|w| + n).$$

This is the new grading we want, as then f will have degree 0. In other words, if we change the grading on V by adding n—this shift is denoted by $V[n]$—, we see that $|v|_{\text{new}} + |w|_{\text{new}} = |f(v \otimes w)|_{\text{new}}$. For example, if f is of degree 2, $|v| = 3$ and $|w| = 4$, then $|f(v \otimes w)| = 7 + 2 = 9$. Shifting the grading by 2, we get $f(v \otimes w) = 9 + 2 = 11 = 5 + 6 = |v| + |w|$. Thus, as a map from $V[n] \otimes V[n] \to V[n]$, f is of degree 0 and thus can be thought of as a multiplication. Note here, that changing the grading is completely superficial. In doing so, we do not alter in the slightest the vector space itself or its elements; indeed, $V \cong V[n]$ as vector spaces. The difference is that $V[n]$ may be a graded algebra, whereas V with the same structures may not, and vice versa. Consider the following example:

Example 7.1. Let $A = k[t]$. This is a graded algebra, with the degree of the polynomials being its nonnegative grading. Now, consider $(k[t])[-1]$. The following table gives the grading of the two vector spaces.

Degree	$k[t]$	$(k[t])[-1]$
-1		1
0	1	x
1	x	x^2
2	x^2	x^3
$\vdots$	$\vdots$	$\vdots$

Note here that $k[t] \cong (k[t])[-1]$ as vector spaces, and (perhaps?) even as (ungraded) algebras. However, as graded algebras they are obviously nonisomorphic, since $(k[t])[-1]$ is not even a graded algebra, since it has is no well-defined multiplication.

We also provide a more interesting geometric example of when we would need to perform this grading "shift."

Example 7.2. Suppose we are dealing with lines in $\mathbb{R}^2$, and want to define a multiplication in terms of intersections. The grading on such a vector space might be the vector-space-dimension of the object, so that lines have degree 1, points of degree 0. We want it so that given ℓ_1 and ℓ_2, $\ell_1 \cdot \ell_2$ has the same degree as ℓ_1 and ℓ_2. Clearly, intersecting two lines has degree -2, so by the above, we need to shift our grading by two. Thus, lines have degree -1; points have degree -2. Hence, taking $\ell_1 \cdot \ell_2$ gives us an element of degree $-2 = |\ell_1| + |\ell_2|$ as desired.

7.2 Algebras, Double Poissons, Lie Algebras, and Poissons

Recall that before, our goal was to describe, explicitly, the map that makes this diagram commute:

$$
\begin{tikzcd}
\{\text{Linear Double Poissons on } T(V) = A\} \arrow[r, "\{-,-\}_{T(V)_{\natural\natural}}"'] \arrow[d, leftrightarrow] & \{\text{Linear Poissons on } A_n\} \arrow[d, leftrightarrow] \\
\{\text{Associative Algebra Structures on } V\} \arrow[r, dashed, "?"'] & \{\text{Lie Algebra Structures on } V_n\}.
\end{tikzcd}
$$

To review, we know the dashed arrow exists since we may traverse the diagram starting from the bottom left node and arriving at the bottom right one. However, to get such a map is complicated—we must first take an associative algebra structure on V, get from it a linear Poisson structure on $T(V)$, which gives us a linear Poisson bracket on $T(V)_n$ that may be restricted to a Lie bracket on V_n—, hence why we want something simpler and just in terms of V and V_n. Next, once we have found such a map, we most importantly want to find some sort of "interpretation" for it. In other words, we want to describe it and explain why it exists without pulling a formula out of thin air and proving that it works; instead we want to study it further and draw some deeper conclusions about the structures on V and V_n and how they are related. More specifically, we want to explain the existence of the arrow in question without having to use the rest of the diagram. First, some setup. Let $V = \text{span}\{x^1, \ldots, x^m\}$, and suppose we have an associative algebra structure on V given by

$$x^\alpha x^\beta = \sum_\gamma a^\gamma_{\alpha\beta} x^\gamma.$$

Given this structure on V, we may define a Lie bracket $[-,-]V_n \otimes V_n \to V_n$ by

$$[x^\alpha_{ij}, x^\beta_{kl}] = \delta_{il}(x^\alpha x^\beta)_{kj} - \delta_{kj}(x^\beta x^\alpha)_{il}.$$

We will not prove that this is a Lie bracket directly, though it can be checked in this way. Thus, we have our arrow—we may check that it is indeed the correct arrow—, and now we find a way to interpret it.

One should immediately note the above bracket's similarity to both the commutator bracket and the Poisson bracket on A_n induced by a double Poisson structure on A by $\{x_{ij}, x_{kl}\} = \{\!\{x, x\}\!\}'_{kj}\{\!\{x, x\}\!\}''_{il}$. Recall that every associative algebra is also equipped with a Lie bracket (the commutator bracket) for free. This will be helpful later. Now, we make the following claim: $V \otimes M_n(k)^* \cong V_n$ as vector spaces, where $v \otimes E^*_{ij} \mapsto v_{ij}$. This is left to the reader to verify. Now, recall that if A and B are algebras, $A \otimes B$ is likewise an algebra. Thus, since V is an algebra, if we defined a multiplication on $M_n(k)^*$, $V \otimes M_n(k)^*$ would also be an algebra. Then, this would induce an algebra structure on V_n; from here we may define a Lie bracket using commutators. Thus, the crux of the problem is finding a compatible algebra structure on $M_n(k)^*$. This is not as difficult as it sounds. Indeed, since $M_n(k)$ is an algebra, and since $M_n(k) \cong M_n(k)^*$ by the map $\text{tr}(E_{ij}, -) = E^*_{ji}$, there is a bijective correspondence between the basis of $M_n(k)$ and its dual basis, under which $E_{ij} \leftrightarrow E^*_{ji}$. Using this correspondence and the multiplication in $M_n(k)$, we may define a multiplication in $M_n(k)^*$:

$$
\begin{aligned}
E^*_{ij}E^*_{kl} &\leftrightarrow E_{ji}E_{lk} = \delta_{il}E_{jk} \leftrightarrow \delta_{il}E^*_{kj} \\
E^*_{kl}E^*_{ij} &\leftrightarrow E_{lk}E_{ji} = \delta_{kj}E_{li} \leftrightarrow \delta_{kj}E^*_{il}.
\end{aligned}
$$

Thus, $V_n \cong V \otimes M_n(k)^*$ becomes an associative algebra, on which we have a Lie bracket via the commutator bracket. In other words, we may define the following bracket on V_n:

$$
\begin{aligned}
[x^\alpha_{ij}, x^\beta_{kl}] := x^\alpha_{ij}x^\beta_{kl} - x^\beta_{kl}x^\alpha_{ij} &\leftrightarrow (x^\alpha \otimes E^*_{ij})(x^\beta \otimes E^*_{kl}) - (x^\beta \otimes E_{kl})(x^\alpha \otimes E^*_{ij}) \\
&= x^\alpha x^\beta \otimes E^*_{ij}E^*_{kl} - x^\beta x^\alpha \otimes E^*_{kl}E^*_{ij} \\
&= x^\alpha x^\beta \otimes \delta_{il}E^*_{kj} - x^\beta x^\alpha \otimes \delta_{kj}E^*_{il} \leftrightarrow \delta_{il}(x^\alpha x^\beta)_{kj} - \delta_{kj}(x^\beta x^\alpha)_{il}
\end{aligned}
$$

as desired. Deriving the bracket this way, we glean much more information about the relationship between V and V_n and their corresponding structure. Notably, we may add the following step in our commutative diagram:

$$
\begin{tikzcd}
\{\text{Linear Double Poissons on } T(V) = A\} \arrow[r, "\{-,-\}_{T(V)_{\natural\natural}}"'] \arrow[d, leftrightarrow] & \{\text{Linear Poissons on } A_n\} \arrow[d, leftrightarrow] \\
\{\text{Associative Algebra Structures on } V\} \arrow[dr] & \{\text{Lie Algebra Structures on } V_n\} \\
 & \{\text{Associative Algebra Structures on } V_n\} \arrow[u, "xy-yx"']
\end{tikzcd}
$$

where we first descend to a (potentially nonunital) associative algebra structure on V_n, and then use the commutator bracket to induce a Lie bracket on V_n. Thus, we have our interpretation of the bracket.

7.3 Another Lie Bracket

There are other ways for associative algebra structures on V to induce Lie brackets on V_n. For example,

$$[x_{ij}^{\alpha}, x_{kl}^{\beta}] := (x^{\alpha}x^{\beta})_{ij} - (x^{\beta}x^{\alpha})_{kl} = -((x^{\beta}x^{\alpha})_{kl} - (x^{\alpha}x^{\beta})_{ij}) = -[x_{kl}^{\beta}, x_{ij}^{\alpha}]$$

is another bracket we could study. While this one is less interesting than the one defined above (for example, it does not make the diagram commute as desired), we may still ask questions about its interpretation, just as we did in the above. What is perhaps trickiest about this bracket is proving its well-definedness. When going back and forth between V and V_n, we need to make sure our bracket is behaving in the way we would like it to and that we do not pick up anything fishy along the way. For example, we verify that the bracket is indeed bilinear, and that it is a Lie bracket. Bilinearity is not difficult to show, partly because we have defined the bracket on a basis and extended bilinearly. But, in order to verify that the bracket is well-defined, we check it anyways:

$$\begin{aligned}
[x_{ij}^{\alpha}, x_{kl}^{\beta} + x_{kl}^{\gamma}] &= [x_{ij}^{\alpha}, x_{kl}^{\beta}] + [x_{ij}^{\alpha}, x_{kl}^{\gamma}] \\
&= (x^{\alpha}x^{\beta})_{ij} - (x^{\beta}x^{\alpha})_{kl} + (x^{\alpha}x^{\gamma})_{ij} - (x^{\gamma}x^{\alpha})_{kl} \\
&= (x^{\alpha}(x^{\beta} + x^{\gamma}))_{ij} - ((x^{\beta} + x^{\gamma})x^{\alpha})_{kl} \\
&= [x_{ij}^{\alpha}, (x^{\beta} + x^{\gamma})_{kl}]
\end{aligned}$$

(note that if we used x_{st}^{γ} instead of x_{kl}^{γ}, this check is trivial). Now, we show that the bracket satisfies the Jacobi identity. Straight computation shows that:

$$\begin{aligned}
[x_{ij}^{\alpha}, [x_{kl}^{\beta}, x_{st}^{\gamma}]] &= (x^{\alpha}x^{\beta}x^{\gamma})_{ij} - (x^{\beta}x^{\gamma}x^{\alpha})_{kl} - (x^{\alpha}x^{\gamma}x^{\beta})_{ij} + (x^{\gamma}x^{\beta}x^{\alpha})_{st} \\
[x_{kl}^{\beta}, [x_{st}^{\gamma}, x_{ij}^{\alpha}]] &= (x^{\beta}x^{\gamma}x^{\alpha})_{kl} - (x^{\gamma}x^{\alpha}x^{\beta})_{st} - (x^{\beta}x^{\alpha}x^{\gamma})_{kl} + (x^{\alpha}x^{\gamma}x^{\beta})_{ij} \\
[x_{st}^{\gamma}, [x_{ij}^{\alpha}, x_{kl}^{\beta}]] &= (x^{\gamma}x^{\alpha}x^{\beta})_{st} - (x^{\alpha}x^{\beta}x^{\gamma})_{ij} - (x^{\gamma}x^{\beta}x^{\alpha})_{st} + (x^{\beta}x^{\alpha}x^{\gamma})_{kl}.
\end{aligned}$$

Hence, the Jacobi identity is satisfied.

In doing these direct checks, we should be able to convince ourselves that this is truly a Lie bracket induced by an associative algebra structure on V. Next, we should try to apply the same line of thinking as above to explain the existence of this bracket. Is there a multiplication on $M_n(k)^*$ that induces an associative algebra structure on V_n that, through the commutator bracket, gives the above bracket? If so, we would have, in $V \otimes M_n(k)^*$,

$$(x^{\alpha} \otimes E_{ij}^*)(x^{\beta} \otimes E_{kl}^*) - (x^{\beta} \otimes E_{kl}^*)(x^{\alpha} \otimes E_{ij}^*),$$

which clearly forces $E_{ij}^* \cdot E_{kl}^* := E_{ij}^*$. Thus, we must check that this is a multiplication; if so, we will have found an "interpretation" for this bracket that is very similar to our explanation of the first bracket we studied. Unfortunately, this is not the case, since this multiplication is obviously not bilinear:

$$E_{ij} * (E_{kl}^* + E_{kl}^*) = E_{ij}^*(2E_{kl}^*) = E_{ij}^* \neq E_{ij}^* + E_{ij}^* = E_{ij}^*E_{kl}^* + E_{ij}^*E_{kl}^*.$$

This is, in part, why we were so careful earlier with checking that this is a Lie bracket—it seems strange that it is not induced by a multiplication structure on $M_n(k)^*$; upon seeing this, one should stop to first check if we have made a mistake, and then try to find an explanation for this oddity. Of course, if V and W are k-vector spaces, not all algebra structures on $V \otimes W$ are induced by individual algebra structures on $V \otimes W$. Indeed, it may be the case that V and W have no algebra structure, but that $V \otimes W$ does. Thus, the bracket may be induced by a multiplication on $V \otimes M_n(k)^*$, where there is no well-defined multiplication on $M_n(k)^*$ that we can recover. Note that $(a_1 \otimes 1)(a_2 \otimes 1)$ is not necessarily in $A \otimes 1$. Thus, we check if the following defines a multiplication on $V \otimes M_n(k)^*$:

$$(v \otimes cE_{ij}^*)(w \otimes c' E_{kl}^*) := cc'(vw \otimes E_{ij}^*).$$

If this is a well-defined multiplication on $V \otimes M_n(k)^*$, then clearly it induces the Lie bracket in question. (I think this multiplication works, but will spend more time trying to break it!)

8 Free Lie Algebras and the Universal Enveloping Algebra

8.1 An Example Of a Graded Algebra

We discuss further our geometric example of a graded algebra. Consider two lines in $\mathbb{R}^2$ given by the kernel of $ax+by$ and $cx+dy$. Recall from linear algebra that $\dim\ker_1 = 2-\mathrm{rank}_1$ and $\dim\ker_2 = 2-\mathrm{rank}_2$, and that

$$\dim\ker_{1\cap 2} = 2-(2-\mathrm{rank}_1)-(2-\mathrm{rank}_1) = \dim\ker_1 + \dim\ker_2 - 2.$$

Thus, if we want our "multiplication" of two lines to be given by the intersection of the lines, clearly we have an operation of degree -2—the dimension of the kernel of the two lines is 1, and the dimension the kernel of their intersection is 0, so the lines have degree 1 whereas their product has degree 0 instead of degree 2. Thus, we shift the grading of the whole space by 2 so that

$$\dim\ker_{1\cap 2} -2 = \dim\ker_1 -2 + \dim\ker_2 -2,$$

which makes sense. Now, our multiplication is of degree 0.

8.2 More on V_n

It turns out that the bracket on V_n we discussed towards the end of the previous section *is* induced by a multiplication on $M_n(k)^*$. First, recall the definition of the bracket:

$$[x_{ij}^\alpha, x_{kl}^\beta] := (x^\alpha x^\beta)_{ij} - (x^\beta x^\alpha)_{kl},$$

which in $V\otimes M_n(k)^*$ corresponds to

$$[x^\alpha\otimes E_{ij}^*, x^\beta\otimes E_{kl}^*] = (x^\alpha x^\beta)\otimes E_{ij}^* - (x^\beta x^\alpha)\otimes E_{kl}^*.$$

The mistake we made was this: $E_{ij}^*E_{kl}^* := E_{ij}^*$ is *not* the same as $XY = X$ for any X, Y in the vector space. Recall that a multiplication on M_n^* is a linear map $m : M_n^*\otimes M_n^* \to M_n^*$; to define such a map, we just need the images of basis vectors. We see that $E_{ij}^*\otimes E_{kl}^* \mapsto E_{ij}^*$ is indeed such a linear map, but that it is not equivalent to $XY = X$ for all $X, Y \in M_n^*$. Suppose $x = E_{ij}^*$, $y = E_{kl}^*$. Clearly,

$$(ax+by)(cx+dy) = acx^2 + adxy + bcyx + bdy^2 = acx + adx + bcy + bdy = (ax+by)(c+d),$$

which is obviously not equal to $(ax+by)$. So we did get a multiplication structure on M_n^* after all! Recall that this gives an algebra structure on $V_n \cong V\otimes M_n(k)^*$, on which we may use the commutator bracket to define a Lie algebra structure on V_n. However, this case is not very interesting, mainly because it has nothing to do with linear double Poisson brackets on $T(V)$ and linear Poisson brackets on $\mathrm{Sym}(V)$, but also because the multiplication on M_n^* is entirely reliant on the standard basis of M_n^*.

8.3 Semidirect Products

In this section, we return to group theory for an introduction to semidirect products. First, recall the definition of a direct product of groups:

Definition 8.1. Let G, H be groups. The direct product of G and H is the group whose underlying set is $G\times H$ and whose multiplication is given by $(g_1,h_1)(g_2,h_2) := (g_1g_2, h_1h_2)$ for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$. It is left to the reader to verify that $G\times H$ with this operation is indeed a group.

Note that both G and H may be embedded into $G\times H$ via the maps $g\mapsto (g,1)$ and $h\mapsto(1,h)$. Moreover, the subgroups $(G,1)$ and $(1,H)$ are normal. For example,

$$(g_1,h)(g,1)(g_1,h)^{-1} = (g_1gg_1^{-1}, hh^{-1}) = (g_1gg_1^{-1},1) \in (G,1).$$

Thus, $G, H \trianglelefteq G \times H$. Let $N, N' \trianglelefteq G$, where G is a group. Because N is normal, there is a G-action by conjugation on N. Restricting this to N', we get $N' \curvearrowright N$, i.e. we have a group homomorphism $\phi : N' \to (N)$ (it is easy to convince oneself that these are the same thing).

Definition 8.2. Let G and H be groups, and suppose $H \curvearrowright G$ by the homomorphism $\phi : H \to (G)$. Then, the semidirect product of G and H, denoted $G \rtimes H$, is defined to be the group with underlying set $G \times H$ and with

$$(g_1, h_1)(g_2, h_2) := (g_1 \phi(h_1) g_2, h_1 h_2)$$

as its binary operation.

It is left to the reader to show that $G \rtimes H$ is indeed a group (Hint: let $(g, h)^{-1} = (\phi(h)^{-1} g^{-1}, h^{-1})$). Note that if $H \curvearrowright G$ trivially, then $G \rtimes H = G \times H$, so semidirect products are indeed a generalization of direct products. Moreover, it is easy to see that G and H are still embedded in $G \rtimes H$ by $(G, 1)$ and $(1, H)$ respectively, but note that only $(G, 1)$ is normal:

$$(x, y)(g, 1)(x, y)^{-1} = (x, y)(g, 1)(\phi(y)^{-1} x^{-1}, y^{-1}) = (x \phi(y) g x^{-1}, 1) \in (G, 1),$$

whereas

$$(x, y)(1, h)(\phi(y)^{-1} x^{-1}, y^{-1}) = (x \phi(y) \phi(y) \phi(h) \phi(y)^{-1} x^{-1}, y h y^{-1}),$$

which may not be in $(1, H)$. We conclude this section with an example:

Example 8.3. Consider the dihedral group of order 8, D_4. Recall that $\mathbb{Z}_4, \mathbb{Z}_2$ are subroups of D_4 but that $D_4 \ncong \mathbb{Z}_4 \times \mathbb{Z}_2$. However, letting $\mathbb{Z}_2 \curvearrowright \mathbb{Z}_4$ by $h \cdot g \mapsto g^{-1}$ for $g \in \mathbb{Z}_4$ and $0 \neq h \in \mathbb{Z}_2$, we have that $D_4 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ (verify this!). In fact, it can be shown (we will not), that if $N \trianglelefteq G$ is normal, and if there exists a map such that $G/N \hookrightarrow G$ so that G/N is isomorphic to a subgroup of G, then $G \cong G/N \rtimes N$. In the case of D_4, $\mathbb{Z}_2$ is normal, so $D_4 \cong D_4/\mathbb{Z}_2 \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$.

8.4 Modules and Representations

Recall that given a group G and a set S, a group action of G on S is a homomorphism $\phi : G \to (S)$, where (S) is the set of bijections $S \to S$. If S has extra structure and is a vector space over k, then a "linear action," or group representation, is a homomorphism $\phi : G \to_k (V) = GL_n(k)$.

If A is a k-algebra, recall that an A-module M is an abelian group $(M, +)$ along with an A-action on M given by an algebra homomorphism $\phi : A \to \mathrm{End}_k(M)$. Thus, we have a bijection between the set of A-module structures on M and the set of homomorphisms from $A \to \mathrm{End}_k(M)$.

Similarly, if $\mathfrak{g}$ is a Lie algebra, a $\mathfrak{g}$-module is a vector space V along with a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{End}_k(V))$ (recall that $\mathrm{End}_k(V)$ is a Lie algebra via the commutator bracket). If $g, h \in \mathfrak{g}$ and $v \in V$, this corresponds to

$$[g, h]v = ghv - hgv.$$

The trivial action on V is given by $g \cdot v := 0$ for all $g \in \mathfrak{g}$, $v \in V$. Moreover, note that $\mathfrak{g}$ becomes a $\mathfrak{g}$-module by the action given by $g \cdot g' := [g, g']$. The Jacobi identity implies that

$$(g_1 \cdot g_2) \cdot g_3 = [g_1, g_2] \cdot g_3 = [[g_1, g_2], g_3] = [g_1, [g_2, g_3]] - [g_2[g_1, g_3]].$$

Now, for algebras, recall that we must distinguish between left and right actions. However, with groups, we see that given a left action $G \curvearrowright S$, we may define a right action in the following way: for all $g \in G$, $s \in S$,

$$s \cdot g := g^{-1} \cdot s,$$

where on the right-hand side of the equation, g^{-1} acts on s using the pre-defined left action we supposedly have. Verifying this is a right action is easy. Obviously, 1 acts as the identity; it is easy to see that for $g, h \in G$,

$$s \cdot (gh) = (gh)^{-1} s = h^{-1} g^{-1} s = h^{-1}(s \cdot g) = (s \cdot g) \cdot h$$

as desired. We may show a similar result for Lie algebra modules. Let $\mathfrak{g}$ be a Lie algebra. It is not difficult to notice that the condition for a right $\mathfrak{g}$-action is as follows: for all $l_1, l_2 \in \mathfrak{g}$ and $v \in V$,

$$v \cdot [l_1, l_2] = (vl_1)l_2 - (vl_2)l_1.$$

Now, suppose that V is a left $\mathfrak{g}$-module. Just as in the group case, the question is: can we make V into a right module given a left action of $\mathfrak{g}$ on V? The answer to the question can be found using the skew-symmetry of the Lie bracket on $\mathfrak{g}$. Since we have a left action of $\mathfrak{g}$ on V, define a right action by $v \cot l = -l \cdot v$ for all $v \in V$ and $l \in \mathfrak{g}$. Then, note that

$$v \cdot [l_1, l_2] = -[l_1, l_2] \cdot v = -(l_1(l_2 v) - l_2(l_1 v)) = -((vl_2)l_1 - (vl_2)l_2) = (vl_1)l_2 - (vl_2)l_1,$$

which by the above is exactly the condition for a right $\mathfrak{g}$-action. Thus, there is no need to distinguish between left and right $\mathfrak{g}$-actions.

8.5 The Free Lie Algebra

Definition 8.4. Let V be a k-vector space. The free Lie algebra $\mathcal{L}(V)$ generated by V is defined by the following process: View $\mathcal{L}(V)$ as a subspace of $T(V)$ and define a bracket on $T(V)$ by

$$[x, y] := x \otimes y - y \otimes x.$$

(what are the quantifiers here? are x and y any elements of $T(V)$? or just generators?) Let $\mathcal{L}(V)$ be the smallest Lie subalgebra of $T(V)$ containing V.

For example,

$$[[x, y], z] = (x \otimes y - y \otimes x) \otimes z - z \otimes (x \otimes y - y \otimes x) \in \mathcal{L}(\mathcal{V}),$$

but $x \otimes y$ is not. The fact that $x \otimes y \notin \mathcal{L}(V)$ is implied by the following proposition.

Proposition 8.5. *Let $\mathcal{L}(V)_n$ denote the free Lie algebra of length n. If $x \in V^{\otimes n}$ and $\mu := [\ldots[[-,-],-]\ldots]$ (there are n slots), then $x \in \mathcal{L}(V)_n$ if and only if $\mu(x) = nx$.*

Before we present a proof, note that this implies $x \otimes y \notin \mathcal{L}(V)_2$ and thus not in $\mathcal{L}(V)$, since $\mu(x \otimes y) = [x, y] = x \otimes y - y \otimes x \neq \frac{1}{2} x \otimes y$.

Proof. The ($\Longleftarrow$) direction is easy: if $\mu(x) = nx$, then $x = \frac{1}{n}\mu(x) \in \mathcal{L}(V)_n$ as desired. The other direction is trickier... □

We work out the $n = 3$ case by computing $\mu([[x, y], z])$. It is not difficult that

$$\mu([[x, y], z]) = \mu(xyz) - \mu(yxz) - \mu(zxy) + \mu(zyx),$$

where we have dropped the tensor notation for convenience. Some computation shows that we get

$$\mu([[x, y], z]) = 3(xyz - yxz - zxy + zyx) = 3[[x, y], z]$$

as desired.

We also have the following nice universal property for free Lie algebras:

Proposition 8.6. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space, both over a field k. Then,*

$$\mathrm{Hom}_k(V, \mathfrak{g}) \cong \mathrm{Hom}_{Lie}(\mathcal{L}(V), \mathfrak{g})$$

□

Note that when $\dim V = 1$, $\mathcal{L}(V) = k$, but when $\dim V = 2$, $\mathcal{L}(V)$ is huge!

8.6 The Universal Enveloping Algebra

We now develop the notion of the Universal Enveloping Algebra. Informally, the universal enveloping algebra connects representations of algebras to representations of Lie algebras. If $\mathfrak{g}$ is a Lie algebra, the universal enveloping algebra of $\mathfrak{g}$, denoted $U_{\mathfrak{g}}$ is the freest associative algebra such that there exists a Lie algebra embedding from $\mathfrak{g} \to U_{\mathfrak{g}}$. Moreover, the Lie algebra representations of $\mathfrak{g}$ are in bijection with the associative algebra representations of $U_{\mathfrak{g}}$; we have $Rep_n(\mathfrak{g}) \cong Rep_n(U_{\mathfrak{g}})$. More formally, the definition is as follows:

Definition 8.7. Let $\mathfrak{g}$ be a Lie algebra over k. The universal enveloping algebra of $\mathfrak{g}$ is defined by

$$U_{\mathfrak{g}} := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]).$$

In other words, it is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two sided ideal generated by $x \otimes y - y \otimes x - [x, y]$.

It is easy to see that $\mathfrak{g}$ embeds into $U_{\mathfrak{g}}$ as a Lie subalgebra via the map $\iota : \mathfrak{g} \to U_{\mathfrak{g}}$, $\iota(x) = x$. Then, $\iota([x, y]) = x \otimes y - y \otimes x = [\iota(x), \iota(y)]$ as desired. Now, we show that for any algebra A, $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A) \cong \mathrm{Hom}_{\mathrm{Alg}}(U_{\mathfrak{g}}, A)$. If $f \in \mathrm{Hom}_{\mathrm{Alg}}(U_{\mathfrak{g}}, A)$, then because associative algebras are also Lie algebras with the commutator bracket, $f \in \mathrm{Hom}_{\mathrm{Lie}}(U_{\mathfrak{g}}, A)$, so $f \circ \iota \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$. Conversely, if $h \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$, then h is also a linear map; thus $h \in \mathrm{Hom}_k(\mathfrak{g}, A)$. By the universal property of tensor algebras, we may extend h to $T(h) \in \mathrm{Hom}_{\mathrm{Alg}}(T(\mathfrak{g}), A)$. Now, the end goal is to show that $T(h)$ may be applied to $U_{\mathfrak{g}}$, thus giving us a homomorphism $U_{\mathfrak{g}} \to A$. To show that $T(h)$ as a function on $U_{\mathfrak{g}}$ is well-defined, we only need to show that

$$T(h)(x \otimes y - y \otimes x - [x, y]) = 0.$$

This is easy:

$$T(h)(x \otimes y - y \otimes x - [x, y]) = h(x)h(y) - h(y)h(x) - h([x, y]) = [h(x), h(y)] - [h(x), h(y)] = 0,$$

where the second equality follows because h is a Lie algebra homomorphism. Thus, we have $T(h) \in \mathrm{Hom}_{\mathrm{Alg}}(U_{\mathfrak{g}}, A)$ as desired. We leave it to the reader to show that these constructions are inverse to each other. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, hook] & A \\
U_{\mathfrak{g}} \arrow[ur] &
\end{tikzcd}$$

Note that in the above diagram, the arrows with source $\mathfrak{g}$ are Lie algebra homomorphisms, whereas the rightmost arrow is an associative algebra homomorphism. It follows that the Lie algebra representations of $\mathfrak{g}$ are in bijection with the associative algebra representations of $U_{\mathfrak{g}}$; we have $\mathrm{Rep}_n(\mathfrak{g}) \cong \mathrm{Rep}_n(U_{\mathfrak{g}})$. Recall that this is exactly the motivation behind the development of the universal enveloping algebra, so we have achieved what we set out to do.

9 More on the Free Lie Algebra and the Universal Enveloping Algebra

9.1 SL_n and $\mathfrak{sl}_n$

In this section, we explain the seemingly mysterious connection between the matrices with determinant 1 and those with trace 0. In other words, we examine the relationship between SL_n and $\mathfrak{sl}_n$. Recall that each Lie algebra corresponds to a linear group. For example,

$$\mathfrak{gl}_n \longleftrightarrow GL_n \subseteq \mathbb{R}^{n^2}$$
$$\mathfrak{sl}_n \longleftrightarrow SL_n$$
$$\mathfrak{sp}_{2n} \longleftrightarrow SP_{2n}$$

(note: GL_n is open in $\mathbb{R}^{n^2}$). Surprisingly, each of the linear groups is quite geometric in nature (e.g. SU_2 is the 3-sphere S^3; U_2 is homeomorphic to $S^3 \times S^1$). It turns out that given a matrix group G, the tangent space of G at I is the Lie algebra of the group. Thus, we want to study this space; by extension we want to study $I + tA$. For any matrix A, note that $I + tA$ will be in GL_n for t sufficiently small, since GL_n is open. Hence, A is in the tangent space of GL_n at I, since $\phi(t) := I + tA$ evaluated at 0 gives I, and $\phi'(0) = A$. Now, recall the following identity:

$$\det(I + tA) = \det(A)t^n + \ldots + \operatorname{tr}(A)t + 1.$$

This will be useful soon. As above $I + tA$ is "locally" in SL_n. To study the tangent space at I, we want to study the derivatives of the paths that go through I. We have

$$\det(A + tI) = 1 = \det(A)t^n + \ldots + \operatorname{tr}(A)t + 1;$$

taking the derivative and evaluating at $t = 0$ gives $0 = \operatorname{tr}(A)$. Thus, the Lie algebra of the special linear group SL_n consists of the trace 0 matrices, which is exactly our previous definition of $\mathfrak{sl}_n$.

9.2 The Universal Enveloping Algebra

Recall the definition of the universal enveloping algebra. If $\mathfrak{g}$ is a Lie algebra, then $U_\mathfrak{g} := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$ for $x, y \in \mathfrak{g}$. Moreover, recall that the key property of the universal enveloping algebra is that $\operatorname{Hom}_{\text{Lie}}(\mathfrak{g}, A) \cong \operatorname{Hom}_{\text{Alg}}(U_\mathfrak{g}, A)$. Letting $A = \mathfrak{gl}_n(k)$, we have

$$\operatorname{Rep}_n(\mathfrak{g}) = \operatorname{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{gl}_n(k)) \cong \operatorname{Hom}_{\text{Alg}}(U_\mathfrak{g}, M_n(k)) = \operatorname{Rep}_n(U_\mathfrak{g}).$$

Example 9.1. We study the following cases—when $\mathfrak{g}$ is the freest and when $\mathfrak{g}$ is the most restricted. In other words, we study $U_\mathfrak{g}$ when $\mathfrak{g}$ is abelian and when it is free. When it is abelian, $[x, y] = 0$ for all $x, y \in \mathfrak{g}$, so $U_\mathfrak{g} = T(\mathfrak{g})/(x \otimes y - y \otimes x)$. In other words, $x \otimes y = y \otimes x$, so $U_\mathfrak{g} = \operatorname{Sym}(\mathfrak{g})$. When $\mathfrak{g} = \mathcal{L}(V)$, $U_\mathfrak{g} = T(V)$, since

$$\operatorname{Hom}_{\text{Lie}}(\mathcal{L}(V), A) \cong \operatorname{Hom}_k(V, A) \cong \operatorname{Hom}_{\text{Alg}}(T(V), A),$$

where the first equality follows from the universal property of the free Lie algebra.

Also, we have the following theorem, often called the "PBW Theorem."

Theorem 9.2 (Poincaré-Birkhoff-Witt). *As vector spaces, Sym*$(\mathfrak{g}) \cong U_\mathfrak{g}$.

While $T(\mathfrak{g})$ is graded, $U_\mathfrak{g}$ is not. But we get a filtration, where $U_\mathfrak{g}^{\leq p}$ denotes the words in $U_\mathfrak{g}$ of length less than or equal to p. Thus, $U_\mathfrak{g}^{\leq p}, U_\mathfrak{g}^{\leq q} \subseteq U_\mathfrak{g}^{\leq p+q}$. Moreover, we see that $\operatorname{Sym}^p(\mathfrak{g})$, the words in $\operatorname{Sym}(\mathfrak{g})$ of length p, is such that $\operatorname{Sym}^p(U_\mathfrak{g}) \cong U_\mathfrak{g}^{\leq p}/U_\mathfrak{g}^{\leq p-1}$. Thus, we have

$$\operatorname{Sym}(\mathfrak{g}) \cong \bigoplus_p U_\mathfrak{g}^{\leq p}/U_\mathfrak{g}^{\leq p-1} \cong U_\mathfrak{g}.$$

It actually turns out that $\operatorname{Sym}(\mathfrak{g}) \cong U_\mathfrak{g}$ as Lie $\mathfrak{g}$-modules, given by the map from $\operatorname{Sym}(\mathfrak{g}) \to U_\mathfrak{g}$ such that

$$x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(x_1 \otimes \cdots \otimes x_n).$$

9.3 Comultiplication on $U_{\mathfrak{g}}$

Recall the definition of comultiplication on an algebra. We may define a compultiplication structure $\Delta : T(V) \to T(V) \otimes T(V)$ given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$ and extending multiplicatively. Thus,

$$\Delta(v_1v_2) = \Delta(v_1)\Delta(v_2) = (v_1 \otimes 1 + 1 \otimes v_1)(v_2 \otimes 1 + 1 \otimes v_2) = v_1v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1v_2.$$

This comultiplication structure on is coassociative, which is defined as follows: Recall the definition of associativity. An operation is associative if $(ab)c = a(bc)$ for all possible elements. In other words, we have the following commutative diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

Thus, just as comultiplication (see the diagram on the right) is given by the "inverse" of the diagram on the left

$$A \otimes A \xrightarrow{\mu} A \qquad\qquad C \xrightarrow{\Delta} C \otimes C$$

coassociativity is defined in an analogous way. In other words, if a comultiplication is coassociative, the following diagram is commutative:

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes \mathrm{id}"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

Now, we expand out $\Delta(v_1v_2v_3)$ and make some observations. Easy computation shows

$$\begin{aligned} \Delta(v_1v_2v_3) = v_1v_2v_3 \otimes 1 + v_1v_2 \otimes v_3 + v_1v_3 \otimes v_2 + v_1 \otimes v_2v_3 \\ + v_2v_3 \otimes v_1 + v_2 \otimes v_1v_3 + v_3 \otimes v_1v_2 + 1 \otimes v_1v_2v_3. \end{aligned}$$

Notice that permuting the tensors in the above expression leaves it unchanged. In other words, it is *cocommutative*: $(12)\Delta = \Delta$. Moreover, note that $\Delta(v_1 \cdots v_n)$ will have 2^n terms, and that the v_i's are always in ascending order.

9.4 The Free Lie Algebra Continued

Recall the following lemma from Section 8:

Proposition 9.3. *Let $\mathcal{L}(V)_n$ denote the free Lie algebra of length n. If $x \in V^{\otimes n}$ and $\mu := [\dots[[-,-],-]\dots]$ (there are n slots), then $x \in \mathcal{L}(V)_n$ if and only if $\mu(x) = nx$.*

In the following we prove this proposition, which requires some work. First, we give the definition of a (p, q)-shuffle.

Definition 9.4. Let $\sigma \in S_{p+q}$. σ is a (p, q)-shuffle if the following conditions are met:

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p); \qquad \sigma(p+1) < \sigma(p+2) < \ldots < \sigma(q).$$

Performing a "riffle" or "waterfall" shuffle on a deck of cards can perhaps make this definition more intuitive to the reader. Note that σ is entirely determined by where $\sigma(1), \ldots, \sigma(p)$ are mapped to.

Thus, there are $\binom{p+q}{p}$ possible (p,q)-shuffles.

Looking more carefully at the $n = 3$ case will help provide more intuition for the following statement. We see that if $n = p + q$,

$$\Delta(v_1 \cdots v_n) = \sum_{(p,q)\text{-shuffles}} (v_{\sigma(1)} \cdots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \cdots v_{\sigma(p+q)})$$

We now state and prove two helpful lemmas.

Lemma 9.5. *Suppose V is a k-vector space and that $x \in \mathcal{L}(V)$. Then $\Delta(x) = x \otimes 1 + 1 \otimes x$, where Δ is the comultiplication on $T(V)$ defined above.*

Proof. It is enough to check that $\Delta([y,z]) = [y,z] \otimes 1 + 1 \otimes [y,z]$ for $y, z \in V$, by the definition of free Lie algebra. Clearly,

$$\begin{aligned}
\Delta([y,z]) &= \Delta(yz - zy) = \Delta(yz) - \Delta(zy) \\
&= \Delta(y)\Delta(z) - \Delta(z)\Delta(y) \\
&= yz \otimes 1 + y \otimes z + z \otimes y + 1 \otimes yz - (zy \otimes 1 + z \otimes y + y \otimes z + 1 \otimes zy) \\
&= yz \otimes 1 + 1 \otimes yz - zy \otimes 1 - 1 \otimes zy = [y,z] \otimes 1 + 1 \otimes [y,z]
\end{aligned}$$

as desired. $\square$

Now define $\gamma : T(V) \to \mathcal{L}(V)$, where $\gamma(v_1, \ldots, v_n) = [\cdots[[v_1, v_2], v_3], \cdots, v_n]$, $\gamma(v) = v$ and $\gamma(1) = 0$. Then, if $x \in T(V)$, $v \in V$, $\gamma(x, v) = [x, v]$. We now present the second lemma we will need:

Lemma 9.6. *Let $x \in T^n(V)$ (i.e. a tensor of length n). Then, $\mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta(x) = nx$.*

Proof. We use induction on n. If $n = 0$, $\Delta(1) = 1 \otimes 1$ so that $\mu \circ (\mathrm{id} \otimes \gamma)(1 \otimes 1) = \mu(1 \otimes 0) = 0$. If $n = 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$. So, we have $\mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta(v) = \mu(v \otimes 0) + \mu(1 \otimes v) = v$. Now, assume the claim is true for some $n-1$, $n \geq 1$. Let $x = v_1 \cdots v_n$. We have

$$\begin{aligned}
\Delta(x) &= \Delta(v_1 \cdots v_n) = \Delta(v_1 \cdots v_{n-1})\Delta(v_n) \\
&= \Big(v_1 \cdots v_{n-1} \otimes 1 + \sum_i a_i \otimes b_i\Big)\Big(v_n \otimes 1 + 1 \otimes v_n\Big) \\
&= v_1 \cdots v_n \otimes 1 + \sum_i a_i v_n \otimes b_i + v_1 \cdots v_{n-1} \otimes 1 + \sum_i a_i \otimes b_i v_n,
\end{aligned}$$

where $b_i \in T(v)^{\geq 1}$. By hypothesis we have

$$\sum_i a_i \gamma(b_i) = \mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta(v_1 \cdots v_{n-1}) = (n-1) v_1 \cdots v_{n-1}.$$

The first equality in the above is a consequence of the following. Recall

$$\Delta(v_1 \cdots v_{n-1}) = \sum_{(p,q)\text{-shuffles}} (v_\sigma(1) \cdots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \cdots v_{\sigma(p+q)}) = v_1 \cdots v_{n-1} \otimes 1 + \sum a_i \otimes b_i$$

for $b_i \in T(V)^{\geq 1}$. Applying $\mathrm{id} \otimes \gamma$ will eliminate the first term after applying μ, so we may ignore it and are left with $(\mathrm{id} \otimes \gamma) \circ \Delta(v_1 \cdots v_{n-1}) = \sum a_i \otimes \gamma(b_i)$. Applying μ gives us the desired expression. Thus,

$$\begin{aligned}
\mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta(x) &= \sum_i a_i v_n \gamma(b_i) + v_1 \cdots v_n + \sum_i a_i \gamma(b_i, v_n) \\
&= \sum_i a_i v_n \gamma(b_i) + v_1 \cdots v_n + \sum_i a_i \gamma(b_i) v_n - a_i v_n \gamma(b_i) \\
&= v_1 \cdots v_n + (n-1) v_1 \cdots v_n = nx
\end{aligned}$$

$\square$

Finally, we prove the following theorem:

Theorem 9.7. *If $x \in T^n(V)$, then the following are equivalent:*

1. $x \in \mathcal{L}(V)$;

2. $\Delta(x) = x \otimes 1 - 1 \otimes x$

3. $\gamma(x) = nx$

Proof. Clearly, (1) $\implies$ (2) follows from Lemma 9.5, and (2) $\implies$ (3) follows from Lemmas 9.5 and 9.6. The final implication (3) $\implies$ (1) follows by rewriting $x = \frac{1}{n}\gamma(x)$. $\square$

10 More on Comultiplication

We first begin with some motivation. By the PBW Theorem, we have that

$$T(V) \cong U(\mathcal{L}(V)) \cong \bigoplus_{p=0}^{\infty} \mathrm{Sym}^p(\mathcal{L}(V)).$$

Moreover, for each p we have an injective map $\mathrm{Sym}^p(\mathcal{L}(V)) \hookrightarrow T(V)$ given by $x_1 \cdots x_p \mapsto \sum_{\sigma \in S_n} \frac{1}{n!}\sigma(x_1 \cdots x_p)$, where $x_i \in \mathcal{L}(\mathcal{V})$. Now how do we describe this decomposition? We turn again to the concept of comultiplication.

10.1 n-fold Comultiplication

Just as associativity allows us to define n-fold multiplication on an algebra, we may analogously define an n-fold comultiplication for a coassociative coalgebra with comultiplication Δ. We do so inductively:

$$\begin{aligned}
&\Delta^{(2)} := \Delta && \Delta^{(2)} : C \to C \otimes C \\
&\Delta^{(3)} := (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta && \Delta^{(3)} : C \to C^{\otimes 3} \\
&\quad\vdots && \\
&\Delta^{(n)} := (\Delta \otimes \mathrm{id} \ldots \otimes \mathrm{id})\Delta^{(n-1)} && \Delta^{(n)} : C \to C^{\otimes n}
\end{aligned}$$

Note here that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ because Δ is coassociative. Recall the comultiplication we defined on $T(V)$, where $\Delta(v) := v \otimes 1 + 1 \otimes v$ for $v \in V$. We see how this n-fold multiplication is realized on $\Delta : T(V) \to T(V) \otimes T(V)$:

$$\begin{aligned}
\Delta^{(2)}(v) &= v \otimes 1 + 1 \otimes v \\
\Delta^{(3)}(v) &= \Delta(v) \otimes 1 + \Delta(1) \otimes v = v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v \\
\Delta^{(4)}(v) &= v \otimes 1 \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 \otimes 1 + 1 \otimes 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes 1 \otimes v \\
&\ \vdots \\
\Delta^{(n)}(v) &= \sum_{i=0}^{n} \sigma^i(v \otimes 1 \otimes \ldots \otimes 1),
\end{aligned}$$

where $\sigma = (1 \cdots n) \in S_n$. Now, recall that $\Delta(ab) = \Delta(a)\Delta(b)$ for $a, b \in T(V)$. Thus, the natural question is this: is $\Delta^{(n)}$ multiplicative as well? We check it for the $n = 3$ case. It is not difficult to see that for $v, w \in V$ $\Delta^{(3)}(v)\Delta^{(3)}(w)$ has 9 terms, since each of $\Delta^{(3)}(v)$ and $\Delta^{(3)}(w)$ have 3 terms. Moreover, it is not difficult to check that

$$\begin{aligned}
\Delta^{(3)}(vw) &= (\Delta \otimes \mathrm{id})\Delta(vw) = (\Delta \otimes \mathrm{id})(vw \otimes 1 + v \otimes w + w \otimes v + 1 \otimes vw) \\
&= vw \otimes 1 \otimes 1 + v \otimes 1 \otimes 1 + w \otimes v \otimes 1 + 1 \otimes vw \otimes 1 + v \otimes 1 \otimes w \\
&\quad + 1 \otimes v \otimes w + w \otimes 1 \otimes v + 1 \otimes w \otimes v + 1 \otimes 1 \otimes vw.
\end{aligned}$$

Combinatorically, the above gives all the possible ways to arrange v and w into $- \otimes - \otimes -$, where 1's are used to denote an empty slot. And since $\Delta^(3)(v)\Delta^{(3)}(w)$ has 9 terms, none of which may be combined, then clearly $\Delta^{(3)}(vw) = \Delta^{(3)}(v)\Delta^{(3)}(w)$. For a less combinatoric argument, take arbitrary $a, b \in$ and see that

$$(\Delta \otimes \mathrm{id})\Delta(ab) = (\Delta \otimes \mathrm{id})(\Delta(a)\Delta(b)) = (\Delta \otimes \mathrm{id})\Delta(a)(\Delta \otimes \mathrm{id})\Delta(b).$$

The last equality in the above follows because why??? We may also use the "prime" notation to prove the above, where $\Delta(a)$ is denoted by $a' \otimes a''$, but this gets confusing quickly. Thus, having proved the multiplicativity of $\Delta^{(n)}$ for $n = 3$, we may use induction for the arbitrary case:

$$\Delta^{(n)}(ab) = (\Delta \otimes \mathrm{id})\Delta^{(n-1)}(ab) = (\Delta \otimes \mathrm{id})\Delta^{(n-1)}(a)\Delta^(n-1)(b) = (\Delta \otimes \mathrm{id})\Delta^{(n-1)}(a)(\Delta \otimes \mathrm{id})\Delta^{(n-1)}(b).$$

Thus, our above computation allows us to see that our n-fold comultiplication Δ may be generalized to arbitrary elements of $T(V)$. It can easily be shown (this should come as no surprise) that $\Delta^{(p)}(v_1 \cdots v_n)$ has p^n terms. Importantly, we note that in $\mathcal{L}(V)$ all of these results will hold, since $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathcal{L}(V)$.

10.2 Another Description of $\mathcal{L}(V)$

Given what we developed above, we may find a new way of looking at $\mathcal{L}(V)$. Define a linear map

$$\psi^p : T(V) \xrightarrow{\Delta^{(p)}} T(V)^{\otimes p} \xrightarrow{\mu} T(V).$$

Note that $\psi^p(x) = px$ for all $x \in \mathcal{L}(V)$, so the free Lie algebra of V is in the eigenspace of ψ^p. Clearly, ψ^1 is the identity on $\mathcal{L}(V)$. Moreover, given $x_1 \cdots x_n \in \mathrm{Sym}^n(\mathcal{L}(V))$, applying ψ^p to the symmetrization $\frac{1}{n!}\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$ of $x_1 \cdots x_n$ gives

$$\psi^p\left(\frac{1}{n!}\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}\right) = \frac{p^n}{n!}\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

We demonstrate this with $p = 3$ and $n = 2$. We calculated $\Delta^{(3)}(vw)$ above, so applying μ gives us $\psi^3(vw) = 6vw + 3wv$. Thus ψ^3 corresponds to $6\,\mathrm{id} + 3(12)$ in kS_n. The symmetrization of vw in kS_n is $\frac{1}{2}\,\mathrm{id} + \frac{1}{2}(12)$, so applying one after the other we get

$$(6\,\mathrm{id} + 3(12))(\tfrac{1}{2}\,\mathrm{id} + \tfrac{1}{2}(12)) = \tfrac{9}{2}\,\mathrm{id} + \tfrac{9}{2}(12) = 9(\tfrac{1}{2}\,\mathrm{id} + \tfrac{1}{2}(12))$$

as desired. Likewise, there is a way to represent ψ^p in kS_n for arbitrary p and n. To summarize, we have the following sequence of isomorphisms, where E_n denotes the eigenspace of ψ^p with eigenvalue p^n:

$$\bigoplus_n \mathrm{Sym}^n(\mathcal{L}(V)) \cong T(V) \cong \bigoplus_n E_n.$$

Lastly, we make the following note: we may also define a "shuffle multiplication" on $T(V)$, given by

$$(v_1 \cdots v_n) \otimes (v_{n+1} \cdots v_{n+m}) \mapsto \sum_{\sigma \in \{(n,m)\text{-shuffles}\}} \sigma(v_1 \cdots v_n v_{n+1} \cdots v_{n+m}).$$

Just as our concatenation multiplication corresponds to our shuffle comultiplication, the above shuffle multiplication corresponds to a "deconcatenation multiplication," where

$$\Delta(vw) := 1 \otimes vw + v \otimes w + vw \otimes 1.$$

Results analogous to the ones proved in the preceding sections may be developed in a similar way.

10.3 Direct Sums and Semidirect Sums of Lie Algebras

Finally we develop the notion of direct and semidirect sums of Lie algebras.

Definition 10.1. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. The direct sum of $\mathfrak{g}$ and $\mathfrak{h}$ is the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ with $\mathfrak{g} \oplus \mathfrak{h}$ (as vector spaces) as its underlying space and bracket given by

$$[(g_1, h_1), (g_2, h_2)] := ([g_1, g_2]_{\mathfrak{g}}, [h_1, h_2]_{\mathfrak{h}}).$$

It is not difficult to show that this is indeed a Lie algebra—it is probably a good exercise to prove it is such.

Now, note that $\mathfrak{g}$ embeds into $\mathfrak{g} \oplus \mathfrak{h}$ by $g \mapsto (g, 0)$; the same can be said for $\mathfrak{h}$ using an analogous argument. Thus, $\mathfrak{g}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{h}$. Moreover,

$$[(g_1, 0), (g_2, h)] \in \mathfrak{g} \oplus 0,$$

for any $(g_2, h) \in \mathfrak{g} \oplus \mathfrak{h}$, so $\mathfrak{g}$ is a Lie ideal of $\mathfrak{g} \oplus \mathfrak{h}$. So is $\mathfrak{h}$. Before defining a semidirect sum of Lie algebras, recall the definition of $\mathrm{Der}(\mathfrak{g})$:

Definition 10.2. A derivation of $\mathfrak{g}$ is a function $\phi \in \mathrm{End}_k(\mathfrak{g})$ such that $\phi([x,y]) = [\phi(x),y] + [x,\phi(y)]$ for all $x,y \in \mathfrak{g}$. The set of all such derivations is denoted $\mathrm{Der}(\mathfrak{g})$. Moreover, as an exercise, show that $\mathrm{Der}(\mathfrak{g})$ is a Lie subalgebra of $\mathrm{End}_k(\mathfrak{g})$ (the Lie algebra of Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g}$).

Now, for the definition of a semidirect sum, we first turn to the following motivation. Just as only G is a normal subgroup in the semidirect product $G \rtimes H$ (recall H is not), we want our semidirect sum to be such that only one of $\mathfrak{g}, \mathfrak{h}$ is an ideal. The question is, how do we define such a bracket? Note that we have

$$[g_1 + h_1, g_2 + h_2] = [g_1, g_2] + [h_1, g_2] - [h_2, g_1] + [h_1, h_2]$$

if $\mathfrak{g} \oplus \mathfrak{h}$ is an internal direct sum of Lie subalgebras. If there is the adjoint action $\mathfrak{h} \curvearrowright \mathfrak{g}$, where $h \cdot g := [h,g] \in \mathfrak{g}$, the first three terms of the right-hand-side equation are all in $\mathfrak{g}$, whereas the last term is in $\mathfrak{h}$. Hence, semidirect sums are defined thusly:

Definition 10.3. Let $\phi : \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$ be a Lie algebra homomorphism. The semidirect sum $\mathfrak{g} \oplus_\phi \mathfrak{h}$ is the Lie algebra with underlying vector space $\mathfrak{g} \oplus \mathfrak{h}$ and bracket given by

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2] + \phi(h_1)(g_2) - \phi(h_2)g_1, [h_1, h_2]).$$

Clearly, if $\mathfrak{h} \curvearrowright \mathfrak{g}$ trivially (i.e. $\phi = 0$), the $\mathfrak{g} \oplus_\phi \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{h}$. Moreover, it is easy to see that while $\mathfrak{g}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g} \oplus_\phi \mathfrak{h}$, only $\mathfrak{g}$ is a Lie ideal, as desired:

$$[(g,0),(g',h)] = ([g,g'],0) \in \mathfrak{g} \oplus 0.$$

As an exercise, determine the universal enveloping algebra of $\mathfrak{g} \oplus \mathfrak{h}$.

11 Derived Poisson Structures

11.1 The Universal Enveloping Algebra of a Direct Sum

In this section, we show that $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes U(\mathfrak{h})$. At first, the natural guess might be to think that $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \oplus U(\mathfrak{h})$. But, this is not true; we may see this by considering $k \oplus k$. In this case, since k is commutative $U(k) = \mathrm{Sym}(k)$, and $\mathrm{Sym}(k) \cong k[x]$, and $\mathrm{Sym}(k \oplus k) \cong k[x, y]$. It is not difficult to see that $k[x, y] \not\cong k[x] \oplus k[y]$. To show that $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes U(\mathfrak{h})$, we invoke a few universal properties. Firstly, we invoke the universal property of universal enveloping algebras. It is not so hard to see that $\mathfrak{g} \oplus \mathfrak{h}$ embeds into $U(\mathfrak{g}) \otimes U(\mathfrak{h})$; there is a Lie algebra homomorphism between the two. This induces an algebra homomorphism between $U(\mathfrak{g} \oplus \mathfrak{h})$ and $U(\mathfrak{g}) \otimes U(\mathfrak{h})$. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{g} \oplus \mathfrak{h} \arrow[r] \arrow[d] & U(\mathfrak{g}) \times U(\mathfrak{h}) \\
U(\mathfrak{g} \oplus \mathfrak{h}) \arrow[ur] &
\end{tikzcd}$$

The diagonal arrow is given by $g \mapsto g \otimes 1$, $h \mapsto 1 \otimes h$, and $gh \mapsto g \otimes h$. Now, we also have that $U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g} \oplus \mathfrak{h}) \hookleftarrow U(\mathfrak{h})$. This induces a bilinear map $U(\mathfrak{g}) \times U(\mathfrak{h}) \to U(\mathfrak{g} \oplus \mathfrak{h})$ under which $g \mapsto g$, $h \mapsto h$, and $g \otimes h \mapsto gh$. On generators, the maps we have found are inverses, which implies an isomorphism between $U(\mathfrak{g} \oplus \mathfrak{h})$ and $U(\mathfrak{g}) \otimes U(\mathfrak{h})$. However, note that it is not necessarily the case that $T(V \oplus W) \cong T(V) \otimes T(W)$.

11.2 Chain Complexes, Homology, and DGA's

Our work with $U(\mathfrak{g})$ begs the question: given a Lie algebra $\mathfrak{g}$, can we find a double Poisson bracket on $U(\mathfrak{g})$? It turns out there are no homogeneous double Poisson brackets on $U(\mathfrak{g})$. If $\{\!\{-,-\}\!\}$ is homogeneous, then the tensors on the left and right sides of the equation below should have the same length:

$$\{\!\{z, xy - yx\}\!\} = \{\!\{z, [x, y]\}\!\}.$$

It is likely this is not the case. Note that we can find a double Poisson bracket on $U(\mathcal{L}(V))$, since this is equal to $T(V)$. But, unfortunately, most interesting Lie algebras are not free. Thus, the word "derived" comes into play. This is related to homotopy, which, recall, is a continuous deformation of one continuous function to another. A couple good examples of homotopies are: (1) the unit disk is homotopic to a point; (2) a coffee cup is homotopic to a doughnut; (3) the punctured plane is homotopic to a circle. In other words, a homotopy between x and y is a sort of topological equivalence between the two objects. Now, with this introduction out of the way, we define a chain complex.

Definition 11.1. A graded vector space $V := \bigoplus_n V_n$ is a chain complex if we have a linear map $d : V_n \to V_{n-1}$ for all n such that $d^2 = 0$.

Definition 11.2. Given a chain complex V, we have the chain of maps below:

$$\cdots \xrightarrow{d_{n+2}} V_{n+1} \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} V_{n-2} \xrightarrow{d_{n-2}} \cdots$$

Because $d^2 = 0$, we have $\ker(d_n) \supseteq \mathrm{im}(d_{n+1})$ for all possible n. Let $H_n := \ker(d_n)/image(d_{n+1})$. The direct sum of all such H_n gives another graded vector space called the homology of $\bigoplus_n V_n$.

Definition 11.3. A differential graded algebra, or DGA for short, is a graded algebra $A = \bigoplus_n A_n$ with a differential d (as above) such that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$.

Note: the $(-1)^{|a|}$ component above is crucial: If we get rid of it, for all a, b,

$$d^2(ab) = d^2(a)b + d(a)d(b) + d(a)d(b) + ad^2(b) = 2d(a)d(b) \neq 0$$

which cannot be the case. Adding it back in, we get

$$d^2(ab) = d^2(a)b + (-1)^{|a|+1}d(a)d(b) + (-1)^{|a|}d(a)d(b) + ad^2(b) = 0.$$

Now suppose A is a DGA algebra. Then it is a graded vector space, so we have the homology of A as a vector space, $H(A) = \bigoplus_n H_n(A)$. It turns out $H(A)$ is a graded algebra: If $a, b \in \ker(d)$, then clearly $d(ab) = 0$. If $a \in \ker(d)$, $b \in A$, then $d(ab) = d(a)b + (-1)^{|a|}ad(b) = (-1)^{|a|}ad(b) \in (d)$, so multiplication in $H(A)$ is well-defined. Thus, $H(A)$ is a graded algebra as desired.

If A, B are DG algebras and if $f : A \to B$ is a DGA homomorphism (i.e. $fd_A = d_B f$), then f descends to a graded algebra homomorphism $f : H(A) \to H(B)$. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "d_A"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_B"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

From here, the definition of a DG isomorphism should be obvious. A quasi-isomorphism between DG algebras A and B is a DGA homomorphism $f : A \to B$ such that $f : H(A) \to H(B)$ is a graded algebra isomorphism.

Any algebra is a DG algebra we force everything to be of degree 0, and let d be the zero map. Then, if B is an associative algebra with this artificial grading and differential,

$$H_n(B) = \begin{cases} B & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Example 11.4. As a simple example, consider the algebra $k[x, y]$ with the trivial grading and differential. Also, consider $k\langle x, y, z\rangle$, where $|x| = |y| = 0$, $|dx| = |dy| = 0$, but $|z| = 1$ and $dz = xy - yx$. Define a map by $x \mapsto x$, $y \mapsto y$, and $z \mapsto 0$. We claim that this is an algebra homomorphism compatible with differentials (check this!). Now, we find $H_0(k\langle x, y, z\rangle)$. To do so, we need to compute $\ker d_0$ and d_1. It is not so difficult to see that $\ker d_0 = k\langle x, y\rangle$, and $d_1 = (xy - yx)$. Thus,

$$H_0(k\langle x, y, z\rangle) = k\langle x, y\rangle/(xy - yx) \cong k[x, y].$$

The higher degree homologies vanish; since this is much harder to prove, we omit it. By (3), $H_0(k[x, y]) = k[x, y]$, so $H_0(k\langle x, y, z\rangle) \cong H_0(k[x, y])$. Therefore, it follows that $H(k\langle x, y, z\rangle) \cong H(k[x, y])$. In other words, $k\langle x, y, z\rangle$ and $k[x, y]$ are quasi-isomorphic.

Now, we state a couple of useful definitions:

Definition 11.5. A DG algebra is semi-free if it is a free algebra when one forgets about the differential. For example, $k\langle x, y, z\rangle$ as given above is a semi-free DG algebra.

Definition 11.6. Suppose A is an algebra and that we may find a semi-free DG algebra (a "semi-free replacement") that is quasi-isomorphic to A (in general, we will want to start with the universal enveloping algebra of some Lie algebra). A Derived Poisson structure on A is a Poisson structure on this semi-free replacement.

11.3 The Chevalley-Eilenberg Chain Coalgebra

In general, there is a way to construct a chain complex for a Lie algebra; such a construction is called the Chevalley-Eilenberg Chain Coalgebra. Recall that given a vector space V, $\mathrm{Sym}(V[1]) \cong \bigwedge^n(V)$ as vector spaces.

Definition 11.7. Suppose $\mathfrak{g}$ is a Lie algebra concentrated in degree 0. The Chevalley-Eilenberg chain coalgebra of $\mathfrak{g}$, denoted $\mathcal{C}(\mathfrak{g};k)$ is given by the ordered pair $\mathrm{Sym}(\mathfrak{g}[1], d_{CE})$, where d_{CE} is the Chevalley-Eilenberg differential. This is given by

$$d_{CE}(g_1 \wedge \ldots \wedge g_n) := \sum_{1\le i<j\le n} (-1)^{i+j-1}[g_i, g_j] \wedge g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots \wedge g_n,$$

where the negative appears from commuting g_i and g_j past the other elements (recall that the hat above the element indicates it is missing from the wedge product).

We check that $d_{CE}^2 = 0$ for the $n = 3$ case:

$$\begin{aligned} d_{CE}^2(X \wedge Y \wedge Z) &= d_{CE}([X,Y] \wedge Z - [x,z] \wedge Y + [Y,Z] \wedge X) \\ &= [[X,Y],Z] - [[X,Z],Y] + [[Y,Z],Z] = 0 \end{aligned}$$

by the Jacobi identity. A similar argument for an arbitrary n works, so the Chevalley-Eilenberg differential is indeed a differential. Moreover the homology of $\mathcal{C}(\mathfrak{g};k)$ is called the Lie algebra homology. As its name suggests, the Chevalley-Eilenberg chain coalgebra also has a comultiplication, which is given by the n-shuffle comultiplication. For example, if we consider $k[x,y]$ with the commutator bracket, the bracket vanishes, so d_{CE} is 0 for $\mathcal{C}(k[x,y];k)$. Moreover, we may learn more about its comultiplication using the following table:

Degree	Element	Δ
0	k	$\Delta(1) = 1 \otimes 1$
1	x, y	$\Delta(x) = x \otimes 1 + 1 \otimes x$
2	$x \wedge y$	$\Delta(x \wedge y) = 1 \otimes (x \wedge y) + x \otimes y - y \otimes x + x \wedge y \otimes 1$

Let $\overline{\mathcal{C}}$ denote the copy of $\mathcal{C}$ in which $1 \mapsto 0$. In other words, the copy of k in $\mathcal{C}$ is eliminated. Now, note that we have the following comultiplication on $\overline{\mathcal{C}}$:

$$\overline{\Delta} : \overline{\mathcal{C}} \xrightarrow{\Delta} C \otimes C \longrightarrow \overline{\mathcal{C}} \otimes \overline{\mathcal{C}}$$

It is not difficult to see that in the example above, $\overline{\Delta}(x) = \pi(x \otimes 1 + 1 \otimes x) = x \otimes 0 + 0 \otimes x = 0$, and $\overline{\Delta}(x \wedge y) = x \otimes y - y \otimes x$. However, clearly x and 0 are of different degrees, so $\overline{\Delta}$ does not preserve the grading. Since Δ is of degree 0, we just need to shift the degree $\overline{\Delta}$ by -1. Thus, we have

$$\overline{\Delta}[-1] : \overline{\mathcal{C}}[-1] \to \overline{\mathcal{C}}[-1] \otimes \overline{\mathcal{C}}[-1],$$

and clearly

$$d_{CE}[-1] : \overline{C\mathcal{C}}[-1] \to \overline{\mathcal{C}}[-1]$$

works as desired. To consider $T(\overline{\mathcal{C}}[-1])$ as a DG algebra, we just need to define a differential on its generators. This we have already done—$T(\overline{\mathcal{C}}[-1])$ has two differentials $\overline{\Delta}[-1]$ and $d_{CE}[-1]$. Lastly, we end with the following theorem:

Theorem 11.8. *Suppose $\mathfrak{g}$ is a Lie algebra. If we construct its Chevalley-Eilenberg chain coalgebra $\mathcal{C}(\mathfrak{g};k)$ and take $T(\overline{\mathcal{C}}[-1])$, then $U_{\mathfrak{g}}$ is quasi-isomorphic to $T(\overline{\mathcal{C}}[-1])$.*

For example, to deal with derived double Poissons on $k[x,y]$, we deal with double Poissons on $k\langle x,y,z\rangle$.