MATH 1070Q: Mathematics for Business and Economics

Lecture Slides

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Topic 7: Counting Techniques and the Sample Space

Reminders and To-Do

- Quiz (Topic 6) at beginning of class.
- Topic 7 will cover sections 7.1 & 7.4 in the text.
- Worksheet 7 at the end of class.
- Quiz (Topic 7) tomorrow morning.

Combinatorics

Counting can be (really) hard. We will learn some techniques.

Definition

Combinatorics is the branch of mathematics which studies counting and arranging elements of certain sets.

One application of combinatorics that we will study in this class is *probability theory*.

The Sample Space

Often we will have an *event* which may have many outcomes.

Definition

The sample space of an event is the set of all its possible outcomes.

We may be interested in counting how many outcomes there are for some event. This is the cardinality of the sample space.

Example 1

Find the sample space for the event of rolling a six-sided die.

Example 2

Find the sample space for the event of picking one finger on your hand at random.

Counting by Making a List

The most natural technique to count something is by *making a list*.

Example

Suppose a mug of coffee is given and there is sugar and cream nearby. How many different variations can this mug of coffee be prepared?

Sample Space for a Sequence of Events

Suppose there are two events E_1 and E_2, which each has their own sample spaces S_1 and S_2. We can determine the sample space for the event in which E_1 happens first and then E_2 happens.

Example

Consider the following two events:

1. Tossing a coin.
2. Rolling a six-sided die.

Find the sample spaces for each event. Then find the sample space of the sequence of events: first tossing a coin, and then rolling a six-sided die.

Question

What is the sample space for first choosing a letter from the word "CAT" and then choosing a number from "578"?

(A) $\{C, A, T, 5, 7, 8\}$.

(B) $\{(C, 5), (A, 7), (T, 8)\}$.

(C) $\{(C, 5), (C, 7), (C, 8), (A, 5), (A, 7), (A, 8), (T, 5), (T, 7), (T, 8)\}$.

(D) $\{(5, C), (7, C), (8, C), (5, A), (7, A), (8, A), (5, T), (7, T), (8, T)\}$

The Direct Product of Sets

What we just did is an example of a set operation called the direct product (or just *product*).

Definition

Let A and B be sets. The direct product of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example

Let $A = \{x, y, z\}$ and $B = \{1, 2, 3, 4\}$. Find...

1. $A \times B$ and $|A \times B|$.
2. $B \times A$ and $|B \times A|$.

Question

How is the number $|A \times B|$ related to $|A|$ and $|B|$?

Cardinality of Direct Products

Properties of Direct Products

1. $|A \times B| = |A| \cdot |B|$.
2. $|A \times B| = |B \times A|$.

Recall the sample space for first tossing a coin and then rolling a die:

$$S = \{(H,1),(H,2),\ldots,(H,6),(T,1),(T,2),\ldots,(T,6)\}$$

We can answer the following question using the properties (1) and (2), instead of listing out all the elements like above:

Question

1. How many possible outcomes are there for first tossing a coin and then rolling a six-sided die?
2. How many possible outcomes are there for first rolling a six-sided die and then tossing a coin?

Counting Outcomes for Three or More Events

We can extend the direct product to three or more sets:

Direct Product of Three Sets

Let A, B, and C be sets. The direct product $A \times B \times C$ is the set

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, \text{ and } c \in C\}.$$

Remark

You could also do $(A \times B) \times C$ or $A \times (B \times C)$. The mechanics we will learn would work the same.

Cardinality of Direct Products

Similarly, we have $|A \times B \times C| = |A| \cdot |B| \cdot |C|$.

The Counting Principle

Because $|A \times B| = |A| \cdot |B|$, we count outcomes of sequences of events by multiplying the outcomes of the individual events:

Counting Principle

The number of possible outcomes for a sequence of k events, where the events have $n_1, n_2, \ldots, n_k$ possible outcomes, is $n_1 \cdot n_2 \cdots n_k$.

Example

An ice cream stand has three flavors of ice cream, two flavors of sauce, and two types of sprinkles. How many different sundaes can be made?

Question

How many outfits can you make if you have 9 shirts, 7 pairs of pants, and 3 pairs of shoes?

(A) 19
(B) 63
(C) 189
(D) 203

Determining Sequences of Events

The difficulty in counting is not in multiplying. . . but in figuring out exactly what the events are.

Some questions will describe a process for a sequence of events without explicitly stating what the number of possible outcomes for each event is.

Example

For example, how many four digit numbers can you create using the digits 1 through 9, if each digit may be used repeatedly? What about if no digit can be used more than once?

For questions like these, we need to determine the number of events, as well as the number of possible outcomes for each event.

Counting with Replacement

Example

How many four-digit numbers can be made using digits 1 through 9?

To solve this question, we need to. . .

1. Determine the individual events.
2. Count the outcomes of each event.
3. Multiply them together to get the total number of possibilities (using counting principle).

Counting without Replacement

Example

How many four-digit numbers can be made using digits 1 through 9 *without duplicate digits*?

To solve this question, we need to...

1. Determine the individual events.
2. Count the outcomes of each event.
3. Multiply them together to get the total number of possibilities (using counting principle).

Question

How many strings of 3 letters can you make using the letters: A, B, C, D, E, F, G, if no letter can be repeated?

(A) 210
(B) 343
(C) 451
(D) 2187